Spaces of locally uniformly bounded functions and applications.

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♣ Dedication ♣

I dedicate this thesis :

My dearest parents **Ernset Réné TAMAGOUA** et **Emilienne TCHUENBOU**,

for the love, affection and valuable advice they constantly give me.

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♣ Contents ♣

♣ Abstract ♣

In this present work we study a locally uniformly bounded function spaces and an application of the properties of these spaces to find the solutions locally uniformly bounded by a second order linear partial differential equation in divergence form in the Sobolev type space $W^{1,2}_{uloc}(\mathbb{R}^d)$. The study of these spaces is motivated by an important step towards the homogenization theory, especially in the resolution of the corrector problem. In this dissertation, we are inspired by existing works insofar as we present in a general way Wiener amalgam spaces and we have restricted ourselves to a particular case of these spaces, that is locally uniformly bounded energy function spaces. Therefore, to solve our equation, We provide in the sense of the distributions an existence and uniqueness result of the weak solution by means of the Caccioppoli's inequality specific to this equation in the Sobolev type space $W^{1,2}_{uloc}(\mathbb{R}^d)$.

Key words

Lebesgue spaces, Amalgams spaces, Sobolev spaces, Caccioppoli's Inequality.

♣ Résumé ♣

Dans ce travail, nous faisons une étude systématique des espaces de fonctions localement uniformément bornées et application des propriétés de ces espaces à la recherche de solutions localement uniformément bornées d'une équation aux dérivées partielles linéaire du second ordre sous forme divergence dans l'espace de type Sobolev $W^{1,2}_{uloc}(\mathbb{R}^d)$. L'étude de ces espaces est motivée par la résolution de problèmes correcteurs en théorie de l'homogénéisation. Dans ce travail, nous nous inspirons des travaux existants dans la mesure où nous présentons de façon générale les espaces amalgames de Wiener et nous nous sommes restreint à un cas particulier de ces espaces : celui des espaces de fonctions à energie localement uniformément bornées. Par suite, dans l'optique de résoudre notre équation, nous établissons au sens des distributions le résultat d'existence et d'unicité de la solution faible de cette équation moyennant l'inégalité de Caccioppoli propre à cette équation dans l'espace de type Sobolev $W^{1,2}_{uloc}(\mathbb{R}^d)$.

Mots clés:

Espaces de Lebesgue, Espaces amalgames, Espaces de Sobolev, Inégalité de Caccioppoli.

♣ Notations ♣

- ☞ d a positive integer.
- ☞ dx denotes the Lebesgue measure on $\mathbb{R}^d$. For all elements $x = (x_i)_{1\leq i\leq d}$ et $y = (y_i)_{1\leq i\leq d}$ de $\mathbb{R}^d$, for each real number $r > 0$ and all elements p et $p^{'}$ de $[1, +\infty]$, we denote by:
- ☞ the scalar product of x and y is defined by $xy = \sum\limits_{i=1}^{d} x_i y_i$.
- ☞ $\|x\| = \left(\sum\limits_{i=1}^{d} x_i^2\right)^{\frac{1}{2}}$ the Euclidean norm of x,
- ☞ $I_x^r = \prod\limits_{i=1}^{d} \left(x_i - \frac{r}{2}, x_i + \frac{r}{2}\right)$,
- ☞ $J_k^r = \prod\limits_{i=1}^{d} \Big[k_i r, (k_i+1)r\Big[$ for all $k = (k_i)_{1\leq i\leq d} \in \mathbb{Z}^d$. J_k^r form a partition of $\mathbb{R}^d$.
- ☞ $p^{'}$ the conjugate exponent of p such that $\frac{1}{p} + \frac{1}{p'} = 1$ with the convention $\frac{1}{+\infty} = 0$.
- ☞ $\check{f}(x) = f(-x)$.

  For any subset E of $\mathbb{R}^d$, we denote by:
- ☞ χ_E its characteristic function,
- ☞ $|E|$ its Lebesgue measure if E is Lebesgue measurable,
- ☞ $\dot{E}$ its interior, $\bar{E}$ its adherence, ∂E the boundary of E.

  For any element X of $\mathbb{R}^d$.
- ☞ $C_c(X)$ the space of continuous functions on X and with compact support,
- ☞ $C^\infty(X)$ the space of continuous functions on X and indefinitely differentiable on X.
- ☞ $L_0(X)$ denotes the complex vector space of equivalence classes modulo the equality λ-almost everywhere of complex functions λ-mesurable on X,
- ☞ $K(X)$ the vector space of continuous complex functions on X with compact support.

♣ General introduction ♣

The amalgam of L^p and l^q on $\mathbb{R}^d$ is a Banach space $(L^p, l^q)(\mathbb{R}^d)$, $(1 \leq p, q \leq +\infty)$ of (classes of) measurable functions on a locally compact abelian group, consisting of L^p functions and having l^q-behaviour at infinity. The idea of considering the amalgam $(L^p, \ell^q)(\mathbb{R}^d)$ as opposed to the Lebesgue space $L^p(\mathbb{R}^d) = (L^p, \ell^p)(\mathbb{R}^d)$ is natural, in that it allows us to distinguish the global behaviour of a mesurable function f from its local behaviour. This idea comes from Nobert Wiener who, in 1926, considered $(L^1, l^2)(\mathbb{R}^d)$ and $(L^2, l^\infty)(\mathbb{R}^d)$ in [22], similarly $(L^\infty, l^1)(\mathbb{R}^d)$ and $(L^1, l^\infty)(\mathbb{R}^d)$. But the first systematic study of these spaces was made in 1975 by Finbaar Holland [10] who established important results in these spaces related to the Fourier transform, several authors have introduced special cases of amalgam spaces during the last decade. These include N. Wiener, T. S. Liu, A. Van Rooij and J. K. Wang [14].

In harmonic analysis, some authors such as: I. Fofana [5], introduced and studied systematically the spaces $(L^p, \ell^q)^\alpha(\mathbb{R}^d)$ which are subspaces of the amalgam spaces $(L^p, \ell^q)(\mathbb{R}^d)$. Among other results he showed that if $1 \leq p \leq q \leq 2$, then the space of Fourier multipliers of $L^p(\mathbb{R}^d)$ in $\ell^q(\mathbb{R}^d)$ is contained in $\left(L^{p'}, \ell^\infty\right)(\mathbb{R}^d)$ with $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{\alpha} = \frac{1}{q} - \frac{1}{p}$. This result was an improvement of the classical result stating that any Fourier multiplier of L^p in ℓ^q is of power p' locally Lebesgue-integrable. In this work, we define a space that is identifiable with the space defined by R. C. Busby and H. A. Smith [3].

The aim of this work is to study the spaces of locally uniformly bounded functions $L^p_{uloc}(\mathbb{R}^d)$ and then to apply the properties of these spaces to the search for locally uniformly bounded solutions of a linear partial differential equation of the second order in divergence form in the Sobolev-type space $W^{1,2}_{uloc}(\mathbb{R}^d)$. Our work is structured as follows:

In the first chapter dedicated to some mathematical results, we recall some essential notions of functional analysis, in particular those relating to the theory of locally compact topological groups and some properties of Lebesgue spaces $L^p(\mathbb{R}^d)$.

In the second chapter, we make a systematic study of the spaces of locally uniformly bounded functions $L^p_{uloc}(\mathbb{R}^d)$. Their study is motivated by the solution of the correcting problems in homogenization. In this section, we give a brief description of the amalgam spaces which are a generalization of these spaces, then we present the properties specific to our study space; we can quote among others Holder's inequality, Young's inequality to quote only a few. We then highlight their relationship and inclusion with other spaces. In the last paragraph, we define the spaces of functions $(L^p_{uloc})^\alpha(\mathbb{R}^d)$, for $1 \leq p \leq \alpha \leq \infty$. We show that for $\alpha \leq +\infty$, the family $(L^p_{uloc})^\alpha(\mathbb{R}^d)$ forms a Banach space chain, the smallest of which is the Lebesgue space $L^\alpha(\mathbb{R}^d)$, and the largest of which is the classical Morrey space which corresponds here to $(L^1_{uloc})^\alpha(\mathbb{R}^d)$. Finally, in the last paragraph, we introduce the Sobolev type spaces $W^{1,2}_{uloc}(\mathbb{R}^d)$ with some properties.

In Chapter 3, we exploit the results of the previous study to solve in the sense of distributions a linear partial differential equation of the second order in divergence form in the Sobolev type space $W^{1,2}_{uloc}(\mathbb{R}^d)$. The aim of this part is to find solutions of this equation in $W^{1,2}_{uloc}(\mathbb{R}^d)$ associated with spaces of functions with locally uniformly bounded energy $L^2_{uloc}(\mathbb{R}^d)$. To this end, we establish the existence and uniqueness theorem of the weak solution which is the main result of this thesis by means of an important mathematical concept: the Caccioppoli inequality specific to our equation in the proof of the said theorem. This inequality is considered as the reciprocal of the Poincaré inequality and is very useful for the study of the regularity of solutions of partial differential equations in general.

⋆ ⋆ CHAPTER ONE ⋆ ⋆

SOME MATHEMATICAL TOOLS

In this chapter, we gather some known notions that will be used throughout this thesis. The first ones we deal with are related to the theory of locally compact topological groups. Subsequently, we introduce Lebesgue spaces with some results. The definitions and properties that are presented can be found in most of the documents in functional analysis, notably [2, 15].

1.1 Recalls on locally compact topological groups

1.1.1 Generalities

Definition 1.1.1. A space X is said to be locally compact if it is separated and if any point x of X has a compact neighbourhood.

If X is locally compact, so is every open O of X. This is because any open of O is an open of X; this brings us back to the definition of compactness by open overlays. If X is locally compact, then so is every closed of X. This is because the intersection of a compact with a closed is still compact, since it is a closed in a compact.

Proposition 1.1. *[15] If X is locally compact and if $X_n \subset X_{n-1} \subset ... \subset X_0 = X$ is a sequence of topological spaces such that X_i is either closed or open in X_{i-1}, then X_n is locally compact.*

Definition 1.1.2. A topological space X is called a Baire space if any countable intersection of dense openings of X is still dense.

Theorem 1.1. [15] Any locally compact space is a Baire space.

Theorem 1.2. [15] Let X be a topological group and $a \in X$ fixed. Then,

1. translations on the left $L_a : x \mapsto ax$ and on the right $R_a : x \mapsto xa$ are homeomorphisms of X in X.

2. the application $x \mapsto x^{-1}$ and the automorphism $x \mapsto axa^{-1}$ are homeomorphisms.

Corollary 1.1. *[15] Let X be a topological group.*

(i) For any open (resp closed) part A of X and any point $a \in X$, the sets aA, Aa and A^{-1} are open (resp closed).

(ii) For any open part O of X and for any part U of X, the sets OU and UO are open.

Theorem 1.3. [15] For a homomorphism of a topological group X into a topological group X' to be continuous, it is necessary and sufficient that it is continuous at a point.

Definition 1.1.3. Let E be a separate topological space. E is said to be homogeneous if for all elements $x, y \in E$, there exists a homeomorphism f such that $f(x) = y$.

Remark 1.1.1. Every topological group is a homogeneous space. Indeed, for all $x, y \in X$, let $a = yx^{-1}$ be the case. The translation L_a is a homeomorphism which applies x to y. Therefore X is a homogeneous space.

Theorem 1.4. Let X be a topological group and H a topological subgroup of X. Then X/H is a homogeneous space.

Definition 1.1.4. A topological group X is locally compact if the underlying space is locally compact.

1.2 Complements

Definition 1.2.1. An open set Ω of $\mathbb{R}^d$ is said to be regular of class C^1 if its $\partial\Omega$ is a regular hypersurface (variety of dimension $d-1$) and if Ω is located on only one side of its boundary.

We define the external normal to the boundary $\partial\Omega$ as being the unit vector $\nu = (\nu_i)_{1 \leq i \leq d}$ normal at any point to the plane tangent to Ω and pointing towards the outside of Ω. Next, we will note by :

$$(u)_U = \int_U u \, dy \qquad \text{et} \qquad (u)_{x,r} = \fint_{B(x,r)} u \, dx = \frac{1}{|B(x,r)|} \int_{B(x,r)} u \, dx.$$

where U is a bounded and connected open of $\mathbb{R}^d$ and $B(x,r) \subset \mathbb{R}^d$ is the open ball of centre x and radius r.

Theorem 1.5. **(Green's formula)**
Let $\Omega \in \mathbb{R}^d$ be a regular open set of class C^1; let ω be a $C^1(\overline{\Omega})^d$ function with bounded support in the closed $\overline{\Omega}$, then it verifies the following Green's formula

$$\int_{\Omega} \text{div}(\omega)(x)\, dx = \int_{\partial\Omega} \omega(x).\nu(x)\, d\sigma_x,$$

where $d\sigma_x$ denotes the surface measure on $\partial\Omega$ and ". "the usual scalar product on $\mathbb{R}^d$ and $\text{div}(\omega) = \partial_{x_1}\omega_1 + \cdots + \partial_{x_d}\omega_d$ is the divergence of the vector $\omega := (\omega_1, \cdots, \omega_d)$.

Let us consider the following formulation of Green's formula.

Corollary 1.2. ***(Integration by parts formula)***
Let $\Omega \in \mathbb{R}^d$ be a regular open set of class C^1; let $u \in C^1(\overline{\Omega})^d$ and $v \in C^1(\overline{\Omega})$ have bounded support in $\overline{\Omega}$, then

$$\int_{\Omega} \nabla v.u\, dx = -\int_{\Omega} v\, div(u) dx + \int_{\partial\Omega} vu.\nu\, d\sigma_x,$$

where $\nabla v = (\partial_{x_1}v_1, ..., \partial_{x_d}v_d)$ represents the gradient of vector v.

Theorem 1.6. (Poincare's inequality)
Let $U \subset \mathbb{R}^d$ be a bounded and connected open set with a ∂U of class C^1. Suppose that $1 \leq p \leq \infty$. Then there exists a constant C depending only on d, p and U such that :

$$\|u - (u)_U\|_{L^p(U)} \leq C\ \|\nabla u\|_{L^p(U)},$$

for any function $u \in W^{1,p}(U)$.

Theorem 1.7. (Poincaré-Wirtinger inequality)
Let $1 \leq p \leq \infty$. There exists a constant C depending only on d and p such that

$$\|u - (u)_{x,r}\|_{L^p(B(x,r))} \leq C\, r\, \|\nabla u\|_{L^p(B(x,r))},$$

for all $B(x,r) \subset \mathbb{R}^d$ and any function $u \in W^{1,p}(B(x,r))$.

1.3 Reflexive spaces

Let E be a Banach space, $E^{"}$ its bidual. We define the following application:

$$\begin{aligned} J : E &\rightarrow E^{"}, \\ x &\rightarrow J(x). \end{aligned}$$

Proposition 1.2. *[2] For all $x \in E$, we have: $||J(x)|| = ||x||$, therefore J is continuous and injective.*

Definition 1.3.1. Let E be a Banach space. E is reflexive if and only if J is surjective.

Remark 1.3.1. If E is reflexive, then $E = E^{"}$ and the topologies $sigma(E^{'}, E^{"})$ and $sigma(E^{'}, E)$ coincide.

Theorem 1.8. (Banach-Alaoglu) [2]

Let E be a Banach space. The closed unit ball of $E^{'}$ denoted $B_{E'}$ is compact for the weak topology '.

Theorem 1.9. (Kakutani)([2])

Let E be a Banach space. Then E is reflexive if and only if the closed unit ball B_E is compact for the weak topology $\sigma(E, E^{'})$.

Lemma 1.1. *Let E be a Banach space, $f_1, ..., f_n \in E^{'}$ and $alpha_1, ..., \alpha_n \in \mathbb{R}$ fixed. The following properties are equivalent:*

1. $\forall \epsilon > 0, \exists x_\epsilon \in E$ *tel que* $||x_\epsilon|| \leq 1$ *et* $|\langle f_i, x_i \rangle - \alpha_i| < \epsilon, \forall i = 1, ..., n.$
2. $\left|\sum_{i=1}^{d} \beta_i \alpha_i\right| \leq \left|\sum_{i=1}^{d} \beta_i f_i\right|, \forall \beta_1, ..., \beta_n \in \mathbb{R}.$

Lemma 1.2. *Let E be a Banach space. Then $J(B_E)$ is dense in $B_{E^{"}}$ for the topology $\sigma(E^{"}, E^{'})$.*

Theorem 1.10. [2] Let E be a Banach. E is reflexive if and only if $E^{'}$ is reflexive.

Proposition 1.3. *[2] Let E be a Banach space. If $E^{'}$ is separable then so is E.*

Proposition 1.4. *(Convergence of sequences)*

- *Let E ba a separable space, and $(\varphi_n)_{n \in \mathbb{N}}$ a sequence of $E^{'}$. If (φ_n) is bounded in $E^{'}$, then there exists (φ_{n_k}) and a sub-sequence (φ_{n_k}) of (φ_{n_k}) such that $(\varphi_{n_k}) \rightarrow \varphi$ for $sigma(E^{'}, E)$.*
- *Let E be a reflexive space, and $(x_n)_n$ a sequence of E. If $(x_n)_n$ is bounded, then there exists an extracted subsequence x_{n_k} of $(x_n)_n$ such that $x_{n_k} \rightarrow x$ for the topology $\sigma(E, E^{'})$.*

Corollary 1.3. *E is reflexive and separable if and only if $E^{'}$ is.*

Theorem 1.11. Let E be a Banach space. E is reflexive if and only if any bounded sequence $(x_n)_{n \in \mathbb{N}}$ of E admits a sub-sequence which converges weakly

Definition 1.3.2. Let E be a Banach space. The space E is said to be uniformly convex if and only if

$$\forall\epsilon>0,\ \exists\delta>0,\ \forall x,y\in E,\ \|x\|_E\leq 1,\ \|y\|_E\leq 1,\ \|x-y\|_E\geq\epsilon\ \Rightarrow\ \left\|\frac{1}{2}(x+y)\right\|_E<1-\delta.$$

Theorem 1.12. [2](Milman-Pettis)
Uniformly convex spaces are reflexive.

1.3.1 Weak and weak* convergence

Till the end, X will denote a Banach space, $X^{'}$ its topological dual and $X^{'}$ the duality bracket between $X^{'}$ and X.

Definition 1.3.3. (Weak convergence)
We say that the sequence $(x_n)\subset X$ converges weakly to $x\in X$ and we note $x_n\rightharpoonup x$ in X if and only if $\langle x^{'},x_n\rangle\longrightarrow\langle x^{'},x\rangle$ for all $x^{'}\in X^{'}$.

Theorem 1.13. Let (x_n) be a sequence of elements of X which converges weakly to x in X, then (x_n) is a bounded sequence in X, i.e. there is a positive constant c independent of n such that $|x_n|_X\leq c$. Moreover, we have

$$\|x\|_X\leq\lim_{n\to\infty}\inf\|x_n\|_X\,.$$

Theorem 1.14. (Weak compactness)
Let X be a reflexive Banach space and (x_n) a bounded sequence of X, then,

- there exists a sub-sequence (x_{n_k}) extracted from $(x_n)_{n\in\mathbb{N}}$ and $x\in X$ such that $x_{n_k}\longrightarrow x$ in X.
- if all sub-sequences $(x_{n_k}$ converge weakly to the same limit x, then the sequence $(x_n)_{n\in\mathbb{N}}$ converges weakly to x in X.

Definition 1.3.4. (Weak* convergence)
The sequence $(x_n^{'})\subset X^{'}$ weak* converges to $x^{'}\in X^{'}$ if and only if $\langle x_n^{'},x\rangle\rightarrow\langle x^{'},x\rangle$ for all $x\in X$.

Theorem 1.15. (Banach-Alaoglu)
Let X be a separable Banach space and $(x_n^{'})$ a bounded sequence of $X^{'}$, then

- there exists a sub-sequence $(x_{n_k}^{'})$ extracted from $(x_n^{'})$ and $x^{'}$ in $X^{'}$ such that $(x_{n_k}^{'})$ converges weakly ' to $x^{'}$ in $X^{'}$.

- If all sub-sequences (x'_{n_k}) converge to the same limit x', then the sequence $(x'_n)_{n\in\mathbb{N}}$ converges weakly ' to x' in X'.

Theorem 1.16. Let X be a Banach space, X' its dual. Let (x_n) and (x'_n) be two sequences of X and X' respectively.

- Let $x_n \rightharpoonup x$ weakly in X, then :

$$\begin{cases} \exists k > 0 \text{ tel que } \forall n \in \mathbb{N} : \|x_n\|_X \leq k, \\ \|x\|_X \leq \lim_{n\to\infty} \inf \|x_n\|_X \,, \end{cases}$$

- Soit $x'_n \rightharpoonup x'$ faiblement dans X', alors :

$$\begin{cases} \exists k > 0 \text{ tel que } \forall n \in \mathbb{N} : \left\|x'_n\right\|_{X'} \leq k, \\ \left\|x'\right\|_{X'} \leq \lim_{n\to\infty} \inf \left\|x'_n\right\|_{X'} \,, \end{cases}$$

- if $x_n \to x$ (strongly in X), then $x_n \rightharpoonup x$ weakly in X.
- if $x'_n \to x'$ (strongly in X'), then $x'_n \rightharpoonup x'$ weakly ' in X'.
- if $x_n \rightharpoonup x$ weakly in X and $x'_n \rightharpoonup x'$ (strongly in X'), then $\left\langle x'_n,\, x_n\right\rangle \rightharpoonup \left\langle x',\, x\right\rangle$.

Proposition 1.5. *Let* $(x_n)_{n\in\mathbb{N}} \subset X$ *and* $(y_n)_{n\in\mathbb{N}} \subset X'$ *such that*

$$x_n \rightharpoonup x \quad \textit{faiblement} \quad \textit{dans} \;\; X,$$
$$y_n \to y \quad \textit{fortement} \quad \textit{dans} \;\; X',$$

then,

$$\lim_{n\to\infty} \langle y_n,\, x_n\rangle_{X',X} = \langle y,\, x\rangle_{X',X} \,.$$

1.4 Lebesgue spaces

Definition 1.4.1. The convolution $f * g$ of two elements f and g of $L_0(\mathbb{R}^d)$ is defined by

$$(f * g)\,(x) = \int_{\mathbb{R}^d} f(x-y)g(y)d(y) \,,$$

in every point $x \in \mathbb{R}^d$.

Let $1 \leq p \leq \infty$. $L^p(\mathbb{R}^d)$ denotes the Lebesgue space on $\mathbb{R}^d$ associated to the Lebesgue's measure, given its usual norm $||.||_p$.

Definition 1.4.2. Let $1 \leq p < \infty$. We denote by $L^p(\mathbb{R}^d)$ the vector space of (classes of) functions $f : \mathbb{R}^d \to \mathbb{R}$ which are measurable and p-th integrable (in the sense of Lebesgue) on $\mathbb{R}^d$. Let $1 \leq p \leq +\infty$. The Lebesgue space $L^p(\mathbb{R}^d)$ is defined by:

$$\|f\|_{L^p(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} |f(x)|^p \, dx\right)^{\frac{1}{p}}.$$

If $p = \infty$, then $L^p(\mathbb{R}^d)$ is the vector space of (classes of) functions $f : \Omega \to \mathbb{R}$ which are measurable and essentially bounded, i.e. $\exists M > 0$ such that $|f(x)| \leq M$ p.p $x \in \Omega$, which we provide with the following norm

$$\|f\|_{L^p(\mathbb{R}^d)} = \text{ess.} \sup_{x\in\mathbb{R}^d} |f(x)| = \inf \left\{M > 0, |f(x)| \leq M, \text{almost everywhere } \; x \in \mathbb{R}^d\right\}.$$

The notion of weak convergence in $L^p(\mathbb{R}^d)$ is defined as follows:

Theorem 1.17. Let $f \in L^p(\mathbb{R}^d)$ and $g \in L^{p'}(\mathbb{R}^d)$. Therefore $(f, g) \in L^1(\mathbb{R}^d)$

- if $1 \leq p < \infty$, then $f_n \to f$ weakly in $L^p(\mathbb{R}^d)$ if

$$\int_{\mathbb{R}^d} (f_n(x), g(x)) \, dx \to \int_{\mathbb{R}^d} (f(x), g(x)) \, dx, \quad \forall g \in L^{p'}(\mathbb{R}^d).$$

- if $p = \infty$, then $f_n \rightharpoonup f$ weakly ' in $L^\infty(\mathbb{R}^d)$ if

$$\int_{\mathbb{R}^d} (f_n(x), g(x)) \, dx \to \int_{\mathbb{R}^d} (f(x), g(x)) \, dx, \quad \forall g \in L^1(\mathbb{R}^d).$$

with the scalar product $(.,.)$ on $\mathbb{R}^d$.

Proposition 1.6. *Let $(u_n)_n$ be a bounded sequence of $L^p(\mathbb{R}^d)$, $1 < p < \infty$, then we can extract from the sequence $(u_n)_n$ a weakly convergent sub-sequence, that is*

$$\exists (u_n)_k, \;\; \exists u \in L^p(\mathbb{R}^d), \;\; \forall \varphi \in L^{p'}(\mathbb{R}^d), \;\; \lim_{k\to\infty} \int_{\mathbb{R}^d} u_{n_k} \, \varphi \, dx = \int_{\mathbb{R}^d} u \, \varphi \, dx.$$

Corollary 1.4. *If $(u_n)_n$ converges weakly to u in $L^p(\mathbb{R}^d)$, then we have*

$$\|u\|_{L^p(\mathbb{R}^d)} \leq \lim_{n\to\infty} \inf \|u_n\|_{L^p(\mathbb{R}^d)}.$$

This result is false in $L^1(\mathbb{R}^d)$ (because this space is not reflexive), on the other hand we have a similar result in $L^{infty}(\mathbb{R}^d)$ provided that we consider the weak topology ' on this space, which is the dual of the separable space $L^1(\mathbb{R}^d)$.

Proposition 1.7. *Let $(u_n)_n$ be a bounded sequence of $L^\infty(\mathbb{R}^d)$. Then we can extract from the sequence $(u_n)_n$ a weakly convergent sub-sequence, that is*

$$\exists \, (u_n)_k, \;\; \exists \, u \in L^\infty(\mathbb{R}^d), \;\; \forall \varphi \in L^1(\mathbb{R}^d), \;\; \lim_{k\to\infty} \int_{\mathbb{R}^d} u_{n_k} \, \varphi \, dx = \int_{\mathbb{R}^d} u \, \varphi \, dx.$$

The product of two weakly convergent sequences does not necessarily converge weakly to the product of limits. On the other hand, if one of the convergences is strong, the result is true.

Proposition 1.8. *Let $1 \leq p,\ q,\ r < \infty$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.*
If $(u_n)_n$ is a sequence of $L^p(\mathbb{R}^d)$ which converges strongly to u in $L^p(\mathbb{R}^d)$, $(v_n)_n$ is a sequence of $L^q(\mathbb{R}^d)$ which converges weakly to v in $L^q(\mathbb{R}^d)$. Then the product sequence $(u_n v_n)$ weakly converges to uv in $L^r(\mathbb{R}^d)$.

Proposition 1.9. *For $1 < p < \infty$, let $(u_n)_n$ a sequence of functions of $L^p(\mathbb{R}^d)$ which weakly converges to u in $L^p(\mathbb{R}^d)$. Let us assume that*

$$\lim_{n\to\infty} \sup \|u_n\|_{L^p(\mathbb{R}^d)} \leq \|u\|_{L^p(\mathbb{R}^d)}\,.$$

Then the sequence $(u_n)_n$ strongly converges to u.

Definition 1.4.3. Let $\varphi \in \mathcal{D}(\mathbb{R}^d)$.

(1) the sequence $(\varphi_n) \subset \mathcal{D}(\mathbb{R}^d)$ converges to φ in $\mathcal{D}(\mathbb{R}^d)$ if there exists a compact s'il existe un compact $k \subset \mathbb{R}^d$ such that

- *supp* $\varphi \subset k$, *supp* $\varphi_i \subset k$, $\forall i \in \mathbb{N}$,
- $D^\alpha \varphi_i \to D^\alpha \varphi$ uniformly on k for any multi-index α

(2) A distribution on $\mathbb{R}^d$ is a linear application $T : \mathcal{D}(\mathbb{R}^d) \to \mathbb{R},\ \ \varphi \to \langle T, \varphi\rangle$ continues on $\mathcal{D}(\mathbb{R}^d)$ and holds $\langle T, \varphi_n\rangle \to \langle T, \varphi\rangle$ in $\mathbb{R}$ for any sequence $(\varphi_n) \subset \mathcal{D}(\mathbb{R}^d)$ such that $\varphi_n \to \varphi$ in $\mathcal{D}(\mathbb{R}^d)$.

Notation 1.4.1. We denote by $\mathcal{D}'(\mathbb{R}^d)$ the space of distributions on $\mathbb{R}^d$.

Definition 1.4.4. Let $f \in \mathcal{D}'(\mathbb{R}^d)$ be a distribution. We define the first derivative of f noted f' or $\frac{df}{dt}$ by the formula

$$\left\langle f', \varphi \right\rangle = -\left\langle f, \varphi' \right\rangle,\ \ \forall \varphi \in \mathcal{D}(\mathbb{R}^d).$$

Proposition 1.10. *For $1 \leq p < +\infty$. For any element (f, g) of $L^p(\mathbb{R}^d) \times L^{p'}(\mathbb{R}^d)$, fg is an element of $L^1(\mathbb{R}^d)$, and $\frac{1}{p} + \frac{1}{p'} = 1$,*

$$||fg|| \leq ||f||_p\ ||g||_{p'} \quad \textit{(Holder's inequality).}$$

Proposition 1.11. *For $1 \leq p < +\infty$, and $g \in L^{p'}(\mathbb{R}^d)$, the application T_g defined by*

$$T_g(f) = \int_{\mathbb{R}^d} f(x)g(x)dx \quad \textit{pour tout}\ \ f \in L^P(\mathbb{R}^d).$$

is a continuous linear form on $L^p(\mathbb{R}^d)$.

Theorem 1.18. [2](Riesz theorem)
Let Ω be an open of $\mathbb{R}^d$, $1 < p < \infty$ and $\varphi \in (L^p(\Omega))^{'}$. Then there exists an unique $u \in L^{p^{'}}(\Omega)$ such that

$$\forall f \in L^p(\Omega),\ \langle \varphi, f \rangle = \int_\Omega uf(x)dx.$$

Moreover,

$$\|u\|_{L^{p'}(\Omega)} = \|\varphi\|_{(L^p(\Omega))'}.$$

Remark 1.4.1. For $1 \leq p < \infty$ such that $\frac{1}{p} + \frac{1}{p^{'}} = 1$. We have the following identification $(L^p)^{'} = L^{p^{'}}$.

Theorem 1.19. The space $L^p(\mathbb{R}^d)$ is reflexive, for every $1 < p < infty$.

Proposition 1.12. *([2]) $L^p(\mathbb{R}^d)$ is separable for $1 \leq p < infty$.*

Theorem 1.20. The set $C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$, $1 \leq p < \infty$.

To prove this theorem, we will recall a result necessary for the demonstration.

Lemma 1.3. *Let $f \in L^1_{loc}(\mathbb{R}^d)$ such that for every*

$$\phi \in C_c(\mathbb{R}^d),\ \int_{\mathbb{R}^d} f(x)\phi(x)dx = 0.$$

Thus, $f = 0$ almost everywhere on $\mathbb{R}^d$.

Proposition 1.13. *Spaces $L^1(\mathbb{R}^d)$ and $L^\infty(\mathbb{R}^d)$ are not reflexives.*

1.4.1 Convolution and regularisation

The convolution product is a classical operation in the case of functions.

Definition 1.4.5. Two functions f and g defined almost everywhere and measurable on $\mathbb{R}^d$ are said to be convolvable if, for almost any $x \in \mathbb{R}^d$, the function $y \mapsto f(x-y)g(y)$ is integrable on $\mathbb{R}^d$.

We then define the convolution product of f and g by the formula

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)dy.$$

Let $p, p^{'}$ be two conjugate exponents. If $f \in L^p$ and $g \in L^{p^{'}}$, then f and g are convoluted. For every $x \in \mathbb{R}^d$, the function that appears under the integral is well integrable since it is the product of $tau_x \breve{f}$ (which belongs to L^p) by g (which is in $L^{p^{'}}$). Thus, $f * g$ is well defined as a function on $\mathbb{R}^d$. Moreover, by the invariance properties of translations and symmetries of the Lebesgue measure, $f * g = g * f$.

Theorem 1.21. (Young's inequality)
Let $1 \leq p, p^{'}, r \leq +\infty$ such that $\frac{1}{p}+\frac{1}{p^{'}}=\frac{1}{r}$+1. Then there exists a constant $C>0$ such that

$$\|f * g\|_r \leq C \;\; \|f\|_p \, \|g\|_{p'}\,, \quad \text{pour tout} f \in L^p(\mathbb{R}^d) \text{ et } g \in L^{p^{'}}(\mathbb{R}^d).$$

Proposition 1.14. *Let $f \in L^1(\mathbb{R}^d)$, $g \in L^p(\mathbb{R}^d)$ et $h \in L^{p^{'}}(\mathbb{R}^d)$. We have*

$$\int_{\mathbb{R}^d} (f * g)\, h \, dx = \int_{\mathbb{R}^d} g \left(\check{f} * h\right) dx \quad \textit{où } \check{f}(x) = f(-x).$$

Proposition 1.15. *Let $f \in C_c(\mathbb{R}^d)$ and $g \in L^1_{loc}(\mathbb{R}^d)$. Then $f * g \in C(\mathbb{R}^d)$.*

Proposition 1.16. *Let $f \in C^k_c(\mathbb{R}^d)$ and $g \in L^1_{loc}(\mathbb{R}^d)$ (k an integer). Therefore*

$$f * g \in C^k(\mathbb{R}^d) \quad \textit{ et } \quad D^\alpha \left(f * g\right) = \left(D^\alpha f\right) * g.$$

In particular if $f \in C^\infty_c(\mathbb{R}^d)$, and $g \in L^1_{loc}(\mathbb{R}^d)$, then $f * g \in C^\infty(\mathbb{R}^d)$ with

$$D^\alpha f = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} ... \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}}, \quad \textit{où} \quad |\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_d \leq k.$$

1.5 ℓ^p spaces, $1 \leq p \leq +\infty$

Definition 1.5.1. Let $1 \leq p \leq q + \infty$, and let $u = (u_n)_{n \in \mathbb{N}}$ be a sequence. We note

$$\|u\|_p = \left(\sum_{n=1}^{+\infty} |u_n|^p \right)^{\frac{1}{p}} \quad \text{si} \quad 1 \leq p < +\infty,$$

and

$$\|u\|_\infty = \sup_{n \in \mathbb{N}} |u_n| \quad \text{si} \quad p = +\infty.$$

We then define the space ℓ^p as the set of sequences u for which the quantity $\|u\|_p$ is finite.

If u, and v are two elements of ℓ^p, $1 \leq p < +\infty$, we check that $\|u+v\|_p \leq \|u\|_p + \|v\|_p$ by using Minkowski's inequality and a passage to the limit. This inequality allows us to show that ℓ^p is a vector space, and that $\|.\|_p$ is a norm on this space.

Proposition 1.17. *ℓ^p is a Banach space, $1 \leq p \leq +\infty$.*

We define $\mathcal{D}$ as the set of almost universally zero sequences with values in $\mathbb{R}$, i.e. zero sequences from a certain rank. Therefore, $u = (u_n)_{n \in \mathbb{N}} \in \mathcal{D}$ if there exists $N \in \mathbb{N}$ for which, for all $n \geq N$, $u_n = 0$.

1.5.1 Topological dual of ℓ^p., $1 \leq p < +\infty$

Let $1 \leq p < \infty$ such that $\frac{1}{p} + \frac{1}{p'} = 1$. Let $v \in \ell^{p'}$. For all $u \in \ell^p$, the real

$$\langle u, v \rangle = \sum_{n \in \mathbb{N}} u_n v_n$$

Holder's inequality ensures that the above series is absolutely convergent and that $|\langle u, v \rangle| \leq \|u\|_{\ell^p} \|v\|_{\ell^{p'}}$. The application

$$\begin{aligned} L_v : \ell^p &\to \mathbb{R}, \\ u &\to \langle u, v \rangle . \end{aligned}$$

Theorem 1.22. Let $1 \leq p < \infty$ et $v \in \ell^{p'}$. The application

$$\begin{aligned} L_v : \ell^p &\to \mathbb{R}, \\ u &\to \langle u, v \rangle . \end{aligned}$$

is a continuous linear form on ℓ^p of norm equal to $\|v\|_{\ell^{p'}}$. Conversely, for any continuous linear form ϕ on ℓ^p, there exists a unique $v \in \ell^{p'}$ such that $\phi = L_v$, and so we have $\|v\|_{\ell^{p'}} = \|\phi\|_{(\ell^p)'}$.

Theorem 1.23. The application L define by :

$$\begin{aligned} L : \ell^p &\to (\ell^p)' \\ v &\to L_v \end{aligned}$$

is a linear and bijective isometry of $\ell^{p'}$ in $(\ell^p)'$.

In particular, the space $(\ell^p)'$ is isometrically isomorphic to the space $\ell^{p'}$. The last theorem above is a representation theorem, which allows to express in a "concrete" way the general form of a continuous linear form on a normed vactor space.

Remark 1.5.1. The topological dual of ℓ^1 is isomorphic to ℓ^∞, but the dual of ℓ^∞ is not isomorphic to ℓ^1. Moreover, the dual of C_0 is isomorphic to ℓ^1.

★ ★ CHAPTER TWO ★ ★

$L^p_{uloc}(\mathbb{R}^d)$ SPACES, $1 \leq p \leq \infty$

Several definitions of amalgams of L^p and ℓ^q, $(1 \leq p \leq q \leq \infty)$ have emerged as a result of research in various fields. The first systematic study of these spaces was made in 1975 by F. Holland [10]. Maria Luisa Torres Desquire [21], mentioned these definitions in chronological order of publication and established their equivalence. In this chapter, we are interested in the study of $L^p_{uloc}(\mathbb{R}^d) = (L^p, \ell^\infty)(\mathbb{R}^d)$.

2.1 $(L^p, \ell^q)(\mathbb{R}^d)$ Spaces

2.1.1 Properties and definitions

Definition 2.1.1. Let $f \in L_0(\mathbb{R}^d)$. For $r > 0$ and $p, q \geq 1$ fixed, we have

$$
{}_r\|f\|_{p,q} = \left[\sum_{k\in\mathbb{Z}^d} \left\|f\chi_{J_k^r}\right\|_p^q\right]^{\frac{1}{q}} \qquad \text{si } 1 \leq p,\ q < \infty
$$

$$
{}_r\|f\|_{\infty,q} = \left[\sum_{k\in\mathbb{Z}^d} \sup_{r>0} \left|f\chi_{J_k^r}\right|^q\right]^{\frac{1}{q}} \qquad \text{si } p = \infty,\ 1 \leq q < \infty
$$

Then the amalgam of L^p and ℓ^q on $mathbbR^d$ is given by:

$$
(L^p, \ell^q)\left(\mathbb{R}^d\right) = \left\{f \in L_0(\mathbb{R}^d),\ {}_r\|f\|_{p,q} < +\infty\right\} \qquad 1 \leq p,\ q \leq \infty.
$$

Some properties of these spaces are given.

Proposition 2.1. *[21] The following inequalities hold true:*

$$
\|f\|_{p,q_2} \leq \|f\|_{p,q_1} \qquad 1 \leq q_1 \leq q_2 \leq \infty,\ 1 \leq p \leq \infty \tag{2.1}
$$

$$
\|f\|_{p_1,q} \leq \|f\|_{p_2,q} \qquad 1 \leq p_1 \leq p_2 \leq \infty,\ 1 \leq q \leq \infty \tag{2.2}
$$

Corollary 2.1. *[21] The following properties are fulfilled*

$$(L^p,\ell^{q_1}) \subseteq (L^p,\ell^{q_2}) \qquad 1\leq q_1\leq q_2\leq\infty,\ 1\leq p\leq\infty \tag{2.3}$$

$$(L^{p_2},\ell^{q}) \subseteq (L^{p_1},\ell^{q}) \qquad 1\leq p_1\leq p_2\leq\infty,\ 1\leq q\leq\infty \tag{2.4}$$

$$(L^p,\ell^{q}) \subseteq L^p\cap L^q \qquad 1\leq q\leq p\leq\infty \tag{2.5}$$

$$L^p\cup L^q \subseteq (L^p,\ell^{q}) \qquad 1\leq p\leq q\leq\infty \tag{2.6}$$

J. Feuto ([6], section 2.1 P.23) established the equivalence between the norms

(i) $\quad {}_r\left\|.\right\|_{p,q} \quad r>0$

(ii) $\quad {}_1\left\|.\right\|_{p,q}$.

Till the end of the work, we will focus on the spaces $L^p_{uloc}(\mathbb{R}^d)$ which is our study space and the main objective of this thesis.

2.2 $L^p_{uloc}(\mathbb{R}^d)$ Spaces

Definition 2.2.1. Let $f\in L_0(\mathbb{R}^d)$. For $r>0$, and $p\geq 1$ fixed, we have

$${}_r\left|\left|f\right|\right|_{p,\infty}=\sup_{x\in\mathbb{R}^d}\left\|f\chi_{I^r_x}\right\|_p . \tag{2.7}$$

Then

$$L^p_{uloc}\left(\mathbb{R}^d\right)=\left\{f\in L_0(\mathbb{R}^d),\ \left|\left|f\right|\right|_{p,\infty}<+\infty\right\}.$$

Definition 2.2.2. Let $\mu\in\nu(\mathbb{R}^d)$ be a measure on $\mathbb{R}^d$. For $q\geq 1$ fixed, we have

$$\left\|\mu\right\|_q=\left[\sum_{k\in\mathbb{Z}^d}\left|\mu\right|(\chi_{J_k})^q\right]^{\frac{1}{q}} \qquad \text{si} \quad 1\leq q<\infty$$
$$\left\|\mu\right\|_\infty=\sup_{x\in\mathbb{R}^d}\left|\mu\right|(\chi_{I_x}) \qquad \text{si} \ \ q=\infty.$$

Then the space of unbounded measures is given by:

$$M_q=\left\{\mu\in\nu(\mathbb{R}^d), \left\|\mu\right\|_q<+\infty\right\}.$$

Proposition 2.2. *(i) $L^p_{uloc}(\mathbb{R}^d)=(L^p,\ell^\infty)(\mathbb{R}^d)$ is a vector subspace of $L_0(\mathbb{R}^d)$.*

(ii) $f\mapsto\left|\left|f\right|\right|_{p,\infty}$ is a norm on $L^p_{uloc}(\mathbb{R}^d)$.

Proof. • Let 0 be the null element of $L_0(\mathbb{R}^d)$. For any mesurable subset E $\subset\mathbb{R}^d$, we have $\left|\left|0\chi_E\right|\right|_p=0$. Hence, $\left|\left|0\right|\right|_{p,\infty}=0$. Thus, $L^p_{uloc}(\mathbb{R}^d)$ has 0 as a zero element and is non-empty.

- Let f and g two elements of $L^p_{uloc}(\mathbb{R}^d)$. For any mesurable subset E $\subset \mathbb{R}^d$, on a:

$$\begin{aligned} ||(f+g)\,\chi_E||_p &= ||f\chi_E + g\chi_E||_p \\ &\leq ||f\chi_E||_p + ||g\chi_E||_p \,. \end{aligned}$$

  In particular, $\forall x \in \mathbb{R}^d$, we have:

$$\begin{aligned} ||(f+g)\,\chi_{I_x}||_p &\leq ||f\chi_{I_x}||_p + ||g\chi_{I_x}||_p \\ &\leq ||f\chi_{I_x}||_{p,\infty} + ||g\chi_{I_x}||_{p,\infty} \,. \end{aligned}$$

  Next, $||(f+g)||_{p,\infty} \leq ||f||_{p,\infty} + ||g||_{p,\infty}$. Therefore, $f+g \in L^p_{uloc}(\mathbb{R}^d)$ and holds $||(f+g)||_{p,\infty} \leq ||f||_{p,\infty} + ||g||_{p,\infty}$.

- let (λ, f) an element of $\mathbb{C}^d \times (L^p, l^\infty)\,(\mathbb{R}^d)$. For any subset E $\subset \mathbb{R}^d$, we have:

$$||(\lambda f)\,\chi_E||_p = ||\lambda\,(f\chi_E)||_p = |\lambda|\,||f\chi_E||_p \,.$$

  In particular, $\forall x \in \mathbb{R}^d$, we have: $||(\lambda f)\,\chi_{I_x}||_p = |\lambda|\,||f\chi_{I_x}||_p$. Next,

$$||\lambda f||_{p,\infty} = |\lambda|\,||f||_{p,\infty} \,.$$

  So, $\lambda.f \in L^p_{uloc}(\mathbb{R}^d)$ and holds $||(\lambda f)||_{p,\infty} = |\lambda|\,||f||_{p,\infty}$.
  Conclusion: From the above, it is clear that $L^p_{uloc}(\mathbb{R}^d)$ is a subvector space of $L_0(\mathbb{R}^d)$ et $f \mapsto ||f||_{p,\infty}$ is a norm on $L^p_{uloc}(\mathbb{R}^d)$.

■

Proposition 2.3. *With $f \longmapsto ||f||_{p,\infty}$; $L^p_{uloc}(\mathbb{R}^d)$ is a Banach space.*

Proof. We know that $\left(L^p_{uloc}(\mathbb{R}^d)\right)$; $||.||_{p,\infty}$ is a normed vector space (according to the proposition 2.2). It remains for us to show that it is complete.
Let $(f_n)_{n\geq 1}$ be a Cauchy sequence in $\left(L^p_{uloc}(\mathbb{R}^d)\right.$; We will show that it converges

1. Let us fix an element $x \in \mathbb{R}^d$. Note that, for any integer n>1,

   (i) $f_n\chi_{I^r_x} \in L^p(\mathbb{R}^d)$,

   (ii) $\forall\, n, n^{'} \in \mathbb{N}^*$, $\left|\left|f_n\chi_{I^r_x} - f_{n'}\chi_{I^r_x}\right|\right|_p \leq\ ||f_n - f_{n'}||_{p,\infty}$.

   Let $\epsilon > 0$ be a real number. Since$(f_n)_{n\geq 1}$ is Cauchy for ${}_r\,||.|_{p,\infty}$, there exists an integer $n_{epsilon} \geq 1$ such that:

$$\forall\, n, n^{'} \in \mathbb{N}^*, \Big(n \geq n_\epsilon \text{ et } n^{'} \geq n_\epsilon\Big)\ \Rightarrow\ ||f_n - f_{n'}||_{p,\infty} < \epsilon.$$

Hence, given (ii), we have:

$$\forall\, n, n^{'} \in \mathbb{N}^*, \left(n \geq n_\epsilon \text{ et } n^{'} \geq n_\epsilon \;\Rightarrow \left|\left|f_n\chi_{I_x^r} - f_{n'}\chi_{I_x^r}\right|\right|_p < \epsilon\right).$$

So $\left(f_n\chi_{I_x^r}\right)_{n\geq 1}$ is a Cauchy sequence in $L^p(\mathbb{R}^d)$. This space being of Banach, $\left(f_n\chi_{I_x^r}\right)_{n\geq 1}$ converges to an element which we note g_x.

$$\text{Posons } g_x = g_x\chi_{I_x^r} \text{ et } ||g_x||_p = \lim_{n\to+\infty} \left|\left|f_n\chi_{I_x^r}\right|\right|_p.$$

2. Let us pose $f = \sup\limits_{x\in\mathbb{R}^d} ||g_x||_p = \sup\limits_{x\in\mathbb{R}^d} \left|\left|g_x\chi_{I_x^r}\right|\right|_p$

   (i) The sequence $(f_n)_{n\geq 1}$ being Cauchy's in $\left(L^p_{uloc}(\mathbb{R}^d);\ ||.||_{p,\infty}\right)$, y is bounded :

$$||f_n||_{p,\,\infty} \leq M < +\infty.$$

   Note that for all $x \in \mathbb{R}^d$, we have:

$$\left|\left|f\chi_{I_x^r}\right|\right|_p = \left|\left|g_x\chi_{I_x^r}\right|\right|_p = ||g_x||_p = \lim_{n\to+\infty} \left|\left|f_n\chi_{I_x^r}\right|\right|_p.$$

   For any finite subset E of $\mathbb{R}^d$, we have:

$$\forall\, n \in \mathbb{N}^*,\ \sup_{x\in E} \left|\left|f_n\chi_{I_x^r}\right|\right|_p \leq\ ||f_n||_{p,\infty} \leq M < +\infty.$$

   As a result, we have:

$$\sup_{x\in E} ||g_x||_p = \lim_{n\to+\infty} \sup_{x\in E} ||f_n\chi_{I_x}||_p \leq M.$$

   i.e. : $\sup\limits_{x\in E} ||g_x||_p \leq M$. This being true for any finite subset of E of $\mathbb{R}^d$, we obtain $\sup\limits_{x\in\mathbb{R}^d} ||g_x||_p \leq M$. i.e. ${}_r\,||f||_{p,\infty} \leq M$. So, $f \in L^p_{uloc}(\mathbb{R}^d)$.

   (ii) Consider a real number $\epsilon > 0$ and a finite subset E of $\mathbb{R}^d$. The sequence $(f_n)_{n\geq 1}$ being Cauchy for $||.||_{p,\infty}$, there exists an integer $n_\epsilon \geq 1$ such that:

$$\forall\, n, m \in \mathbb{N}^*, \left(n \geq n_\epsilon \text{ et } m \geq n_\epsilon \;\Rightarrow\; {}_r\,||f_n - f_m||_{p,\infty} < \epsilon\right).$$

   Now, for all elements n and m of $\mathbb{N}^*$ we have

$$\sup_{x\in E} \left|\left|f_n\chi_{I_x^r} - f_m\chi_{I_x^r}\right|\right|_p = \sup_{x\in E} \left|\left|(f_n - f_m)\,\chi_{I_x^r}\right|\right|_p \geq\ {}_r\,||f_n - f_m||_{p,\infty}.$$

   Thus we have,

$$\forall\, n, m \in \mathbb{N}^*, [n \geq n_\epsilon \text{ et } m \geq n_\epsilon] \;\Rightarrow\; \sup_{x\in E} \left|\left|f_n\chi_{I_x^r} - f_m\chi_{I_x^r}\right|\right|_p < \epsilon. \tag{2.8}$$

Moreover, we know that for any element x of $\mathbb{R}^d$, $\left(f_n\chi_{I_x^r}\right)_{m\geq 1}$ converges in $L^p(\mathbb{R}^d)$ vers $g_x = f\chi_{I_x^r}$. Therefore, for any integer $n \geq 1$ and any element x of $\mathbb{R}^d$, the sequence $\left(f_n\chi_{I_x^r} - f_m\chi_{I_x^r}\right)$ converges in $L^p(\mathbb{R}^d)$ to $\left(f_n\chi_{I_x^r} - f\chi_{I_x^r}\right)$ and therefore we have:

$$\forall\, n \in \mathbb{N}^*, \forall\, x \in \mathbb{R}^d \text{on a: } \lim_{m\to\infty} \left|\left|f_n\chi_{I_x^r} - f_m\chi_{I_x^r}\right|\right| = \left|\left|f_n\chi_{I_x^r} - f\chi_{I_x^r}\right|\right|.$$

Thus, by stretching $m \to \infty$ in the relation (2.8), we obtain:

$$\forall\, n \in \mathbb{N}^*, n \geq n_\epsilon \;\Rightarrow\; \sup_{x\in E} \left|\left|f_n\chi_{I_x^r} - f\chi_{I_x^r}\right|\right|_p \leq \epsilon.$$

E being any finite subset of $\mathbb{R}^d$, we have in fact

$$\forall\, n \in \mathbb{N}^*, n \geq n_\epsilon \;\Rightarrow\; \sup_{x\in \mathbb{R}^d} \left|\left|f_n\chi_{I_x^r} - f\chi_{I_x^r}\right|\right| < \epsilon.$$

i.e.

$$\forall\, n \in \mathbb{N}^*, n \geq n_\epsilon \;\Rightarrow\; {}_r\left|\left|f_n - f\right|\right|_{p,\infty} \leq \epsilon.$$

Thus $(f_n)_{n\geq 1}$ converges to f in $\left(L^p_{uloc}(\mathbb{R}^d);\; ||.||_{p,\infty}\right)$.
Therefore, $\left(L^p_{uloc}(\mathbb{R}^d);\; ||.||_{p,\infty}\right)$ is complete.

■

2.2.1 Inclusive and unequal relationships

Proposition 2.4. *Let* $1 \leq p_1,\, p_2,\; r \leq \infty$.
If $p_1 < p_2$,

$$_r\left|\left|f\right|\right|_{p_1,\infty} \leq\; _r\left|\left|f\right|\right|_{p_2,\infty} \cdot r^{d\left(\frac{1}{p_1}-\frac{1}{p_2}\right)} \qquad \textit{pour tout } \; f \in L_0(\mathbb{R}^d). \tag{2.9}$$

And next,

$$L^{p_2}_{uloc}(\mathbb{R}^d) \subset L^{p_1}_{uloc}(\mathbb{R}^d). \tag{2.10}$$

Proof. Let $1 \leq p_1 < p_2 \leq \infty$ and $f \in L_0(\mathbb{R}^d)$. Note that $1 < \frac{p_1}{p_2} \leq \infty$. Thus, by applying the Holder inequality, we have for any subset E of $\mathbb{R}^d$, measurable and with finite measure.

$$\begin{aligned}
\int_E |f(x)|^{p_1}\, dx \quad &\leq \quad |||f(x)|^{p_1}||_{\frac{p_2}{p_1}}\, ||\chi_E||_{\left(\frac{p_2}{p_1}\right)'} \qquad \text{avec} \left(\frac{p_2}{p_1}\right) + \left(\frac{p_2}{p_1}\right)' = 1,\\
&= \quad ||f\chi_E||_{p_2}^{p_1}\, |E|^{1-\left(\frac{p_2}{p_1}\right)},\\
\text{donc} \quad ||f\chi_E||_{p_1} \quad &\leq \quad ||f\chi_E||_{p_2}\, |E|^{\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}.
\end{aligned}$$

In particular, $\forall x \in \mathbb{R}^d$, we have:

$$\left|\left|f\chi_{I_x^r}\right|\right|_{p_1} \leq \left|\left|f\chi_{I_x^r}\right|\right|_{p_2} \cdot r^{d\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}. \tag{2.11}$$

By using (2.11), we get $\forall x \in \mathbb{R}^d$,

$$\left|\left|f\chi_{I_x^r}\right|\right|_{p_1} \leq \ _r\left|\left|f\chi_{I_x^r}\right|\right|_{p_2} \cdot r^{d\left(\frac{1}{p_1}-\frac{1}{p_2}\right)} \leq \ _r\left|\left|f\right|\right|_{p_2,\infty}.$$

Next, we have:

$$_r\left|\left|f\right|\right|_{p_1,\infty} = \sup_{x\in\mathbb{R}^d}\left|\left|f\chi_{I_x^r}\right|\right|_{p_1} \leq \ _r\left|\left|f\right|\right|_{p_2,\infty} \cdot r^{d\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}.$$

■

Remark 2.2.1. Let f be an element of $L_0(\mathbb{R}^d)$.

- if $||f||_\infty = 0$ then $f = 0$ and next, $\forall x \in \mathbb{R}^d$, $\left|\left|f\chi_{I_x^r}\right|\right|_\infty = 0$ and thus $||f||_{\infty,\infty} = \sup_{x\in\mathbb{R}^d}\left|\left|f\chi_{I_x^r}\right|\right|_\infty = 0 = \left|\left|f\chi_{I_x^r}\right|\right|_\infty$.
- if $||f||_\infty > 0$ alors, pour tout élément $\gamma \in (0, ||f||_\infty)$, nous avons:

  $$0 < \left|\left\{x \in \mathbb{R}^d, |f| > \gamma\right\}\right| = \left|\bigcup_{k\in\mathbb{Z}^d}\left\{x \in J_K^r, ||f|| > \gamma\right\}\right|.$$

  Next, $\exists k \in \mathbb{Z}^d$ such that $0 < |\{x \in I_x^r, |f(x)| > \gamma\}|$ and therefore

  $$\gamma < \left|\left|f\chi_{I_x^r}\right|\right|_\infty \leq ||f||_\infty.$$

  That means $\gamma < \ _r||f||_{\infty,\infty} \leq ||f||_\infty$, hence $_r||f||_{\infty,\infty} = ||f||_\infty$. Thus, in all cases, normal vector spaces $\left(\left(L^p, \ell^p\right)(\mathbb{R}^d); \ _r||.||_{p,p}\right)$ and $\left(\left(L^p\right)(\mathbb{R}^d); ||.||_p\right)$ coincide.

Corollary 2.2. *Let $1 \leq p \leq \infty$. Then,*

$$L^p(\mathbb{R}^d) \cup L^\infty(\mathbb{R}^d) \subset L^p_{uloc}(\mathbb{R}^d). \tag{2.12}$$

Proof. Let $1 \leq p \leq \infty$. We have:

$$L^p(\mathbb{R}^d) = \left(L^p, \ell^p\right)(\mathbb{R}^d) \subset L^p_{uloc}(\mathbb{R}^d).$$

$$L^\infty(\mathbb{R}^d) = \left(L^\infty, \ell^\infty\right)(\mathbb{R}^d) \subset L^p_{uloc}(\mathbb{R}^d).$$

Thus, $L^p(\mathbb{R}^d) \cup L^\infty(\mathbb{R}^d) \subset L^p_{uloc}(\mathbb{R}^d)$. ■

Proposition 2.5. *Let 1$\leq p_1$, p_2, $p \leq \infty$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} \leq 1$. Then for any element f, and g of $L_0(\mathbb{R}^d)$,*

$$_r||fg||_{p,\infty} \leq \ _r||f||_{p_1,\infty} \ _r||g||_{p_2,\infty}. \tag{2.13}$$

Proof. Let f and g be two elements of $L_0(\mathbb{R}^d)$. We can assume ${}_r\,||fg||_{p,\infty} \neq 0$. As a result, ${}_r\,||f||_{p_1,\infty} < \infty$ et ${}_r\,||f||_{p_2,\infty} < \infty$ because otherwise the inequality is trivial. Let r be a strictly positive real. For any subset $E \subset \mathbb{R}^d$, by virtue of Holder's inequality, we have: ${}_r\,||fg\chi_E||_p \leq {}_r\,||f\chi_E||_{p_1}\, {}_r\,||g\chi_E||_{p_2}$. In particular, $\forall x \in \mathbb{R}^d$, we have:

$$ {}_r\left|\left|fg\chi_{I_x^r}\right|\right|_p \leq {}_r\left|\left|f\chi_{I_x^r}\right|\right|_{p_1}\, {}_r\left|\left|g\chi_{I_x^r}\right|\right|_{p_2}. $$

Next,

$$ \begin{aligned} {}_r\,||fg||_{p,\infty} &= \sup_{x\in\mathbb{R}^d} \left|\left|fg\chi_{I_x^r}\right|\right|_{p,\infty} \\ &\leq \sup_{x\in\mathbb{R}^d} \left|\left|f\chi_{I_x^r}\right|\right|_{p_1,\infty} \sup_{x\in\mathbb{R}^d} \left|\left|g\chi_{I_x^r}\right|\right|_{p_2,\infty} \\ &\leq {}_r\,||f||_{p_1,\infty}\; {}_r\,||g||_{p_2,\infty}. \end{aligned} $$

■

Proposition 2.6. *(Holder's inequality)*
Let $1 \leq p,\ q,\ p^{'},\ q^{'} \leq \infty$ *such that* $\frac{1}{p} + \frac{1}{p^{'}} = 1$ *and* $\frac{1}{q} + \frac{1}{q^{'}} = 1$. *We have:*

$$ \int_{\mathbb{R}^d} |f(x)g(x)|\, dx \leq {}_r\,||f||_{p,q}\; {}_r\,||g||_{p^{'},q^{'}}. \tag{2.14} $$

For all $f \in (L^p, \ell^q)(\mathbb{R}^d)$, $g \in \left(L^{p^{'}}, \ell^{q^{'}}\right)(\mathbb{R}^d)$.

Proof. Let $f \in (L^p, \ell^q)(\mathbb{R}^d)$ and $g \in \left(L^{p^{'}}, \ell^{q^{'}}\right)(\mathbb{R}^d)$. From Holder's inequality for L^p spaces, we have:

$$ \forall k \in \mathbb{Z}^d, \int_{J_k^r} |f(x)g(x)|\, dx \leq \left|\left|f\chi_{J_k^r}\right|\right|_p \left|\left|f\chi_{J_k^r}\right|\right|_{p'} $$

next,

$$ \int_{\mathbb{R}^d} |f(x)g(x)|\, dx = \sum_{k\in\mathbb{Z}^d} \int_{J_k^r} |f(x)g(x)|\, dx \leq \sum_{k\in\mathbb{Z}^d} \left|\left|f\chi_{J_k^r}\right|\right|_p \left|\left|g\chi_{J_k^r}\right|\right|_{p'}. \tag{2.15} $$

<u>1st case</u>: $1 < q < +\infty$

From (2.15) and Holder's inequality for series, we get:

$$ \int_{\mathbb{R}^d} |f(x)g(x)|\, dx \leq \left[\sum_{k\in\mathbb{Z}^d} \left|\left|f\chi_{J_k^r}\right|\right|_p^q\right]^{\frac{1}{q}} \left[\sum_{k\in\mathbb{Z}^d} \left|\left|g\chi_{J_k^r}\right|\right|_{p'}^{q'}\right]^{\frac{1}{q'}} = {}_r\,||f||_{p,q}\; {}_r\,||g||_{p',q'}. $$

<u>2nd case</u>: $q = 1$

From (2.15), we have:

$$ \int_{\mathbb{R}^d} |f(x)g(x)|\, dx \leq \left[\sum_{k\in\mathbb{Z}^d} \left|\left|f\chi_{J_k^r}\right|\right|_p\right] \sup_{r>0} \left|\left|g\chi_{J_k^r}\right|\right|_{p'}. $$

Note that for any element k of $\mathbb{Z}^d$,

$$||g\chi_{J_k^r}||_{p'} = \left|\left|g\chi_{J^r_{x(k)}}\right|\right|_{p'} \quad \text{où } \; x(k) = \left(k_1 r + \frac{r}{2}, k_2 r + \frac{r}{2}, ..., k_d r + \frac{r}{2}\right).$$

next,

$$\sup_{x\in\mathbb{Z}^d} ||g\chi_{J_k^r}||_{p'} = \sup_{x\in\mathbb{Z}^d} \left|\left|g\chi_{J^r_{x(k)}}\right|\right|_{p'} \leq \sup_{x\in\mathbb{R}^d} ||g\chi_{J_k^r}||_{p'} = {}_r\, ||g\chi_{J_k^r}||_{p',\infty}.$$

Thus

$$\int_{\mathbb{R}^d} |f(x)g(x)|\, dx \leq {}_r\, ||f||_{p,1}\; {}_r\, ||g||_{p',\infty}.$$

<u>3[rd] case</u>: $q = +\infty$

Based on the above, we have:

$$\begin{aligned} \int_{\mathbb{R}^d} |f(x)g(x)|\, dx &= \int_{\mathbb{R}^d} |g(x)f(x)|\, dx \\ &\leq {}_r\, ||g||_{p',1}\; {}_r\, ||f||_{(p')',\infty} \\ &= {}_r\, ||f||_{p,\infty}\; {}_r\, ||g||_{p',1}. \end{aligned}$$

■

Proposition 2.7. *For any measure $\mu \in M_q(\mathbb{R}^d)$, we have:*

$$\|\mu\|_p \leq \|\mu\|_q \qquad (1 \leq q \leq p \leq \infty). \tag{2.16}$$

Proof. Let $\mu \in M_q(\mathbb{R}^d)$.

(•) if $p = \infty$, then

$$\|\mu\|_\infty = \sup_{x\in\mathbb{R}^d} |\mu|(\chi_{I_x}) \leq \left(\sum_{k\in\mathbb{Z}^d} |\mu|(\chi_{I_k})^q\right)^{\frac{1}{q}} = \|\mu\|_q.$$

(•) if $1 \leq q \leq p < \infty$, then

$$\|\mu\|_p = \left(\sum_{k\in\mathbb{Z}^d} |\mu|(\chi_{J_k})^p\right)^{\frac{1}{p}} \leq \left(\sum_{k\in\mathbb{Z}^d} |\mu|(\chi_{J_k})^q\right)^{\frac{1}{q}} = \|\mu\|_q.$$

■

Theorem 2.1. From the above, we have the following assertions:

$$L^p(\mathbb{R}^d) \subseteq \left(L^1, \ell^p\right)(\mathbb{R}^d) \cap (L^p, \ell^\infty)(\mathbb{R}^d) \quad \text{si} \quad 1 < p < \infty. \tag{2.17}$$

$$L^p(\mathbb{R}^d) \subseteq (L^p, \ell^q)(\mathbb{R}^d) \cap \left(L^1, \ell^p\right)(\mathbb{R}^d) \quad \text{si} \quad 1 < p < q < \infty. \tag{2.18}$$

$$\left(L^p, \ell^1\right)(\mathbb{R}^d) \cap (L^\infty, \ell^p)(\mathbb{R}^d) \subset L^p(\mathbb{R}^d) \quad \text{si} \quad 1 < p < \infty. \tag{2.19}$$

$$(L^p, \ell^q)(\mathbb{R}^d) \cap \left(L^q, \ell^1\right)(\mathbb{R}^d) \subset L^q(\mathbb{R}^d) \quad \text{si} \quad 1 < q < p < \infty. \tag{2.20}$$

$$L^p(\mathbb{R}^d) \subset L^p_{uloc}(\mathbb{R}^d) \quad \text{si} \quad 1 \leq p < \infty. \tag{2.21}$$

Proof. These are direct consequences of Proposition 2.1, Corollary 2.1. ∎

Corollary 2.3. *The following inclusions are fulfilled:*

$$\left(L^p, \ell^1\right)(\mathbb{R}^d) \subset L^p(\mathbb{R}^d) \quad si \quad 1 < p \leq \infty. \tag{2.22}$$

$$L^p(\mathbb{R}^d) \subset \left(L^p, \ell^\infty\right)(\mathbb{R}^d) \quad si \quad 1 \leq p \leq \infty. \tag{2.23}$$

$$L^\infty(\mathbb{R}^d) \subset L^1_\infty(\mathbb{R}^d). \tag{2.24}$$

$$\left(L^\infty, \ell^q\right)(\mathbb{R}^d) \subset L^q(\mathbb{R}^d) \quad si \quad 1 \leq p < \infty. \tag{2.25}$$

Proof. This is a direct consequence of Proposition 2.1, Corollary 2.2. ∎

Remark 2.2.2. From the above, we find that:

$$\left(L^\infty, \ell^1\right)(\mathbb{R}^d) \subseteq \left(L^p, \ell^q\right)(\mathbb{R}^d) \subseteq L^1_{uloc}(\mathbb{R}^d) \quad (1 \leq p,\ q \leq +\infty). \tag{2.26}$$

In other words, $\left(L^\infty, \ell^1\right)(\mathbb{R}^d)$ is the smallest and $L^1_{uloc}(\mathbb{R}^d)$ is the largest of the amalgam spaces. Similarly, $M_1(\mathbb{R}^d)$ is the smallest and $M_\infty(\mathbb{R}^d)$ is the largest of the spaces $M_q(\mathbb{R}^d)$ $(1 < q < +\infty)$

Let $f \in \left(L^1, \ell^\infty\right)(\mathbb{R}^d)$ et $m \in M_\infty(\mathbb{R}^d)$. The measure $fm \in M_\infty(\mathbb{R}^d)$ is such that

$$\int dfm = \int fdm \quad et \quad \|f\|_{1,\infty} = \|fm\|_\infty .$$

Hence, $f \mapsto fdm$ is a natural integration of $L^1_{uloc}(\mathbb{R}^d)$ into $M_\infty(\mathbb{R}^d)$. In this sense, we have:

$$L^1_{uloc}(\mathbb{R}^d) \subset M_\infty(\mathbb{R}^d), \qquad 1 \leq q \leq \infty. \tag{2.27}$$

Note that for $1 \leq p \leq \infty$, we have:

$$\left(L^p, \ell^\infty\right)(\mathbb{R}^d) \subset L^1_{uloc}(\mathbb{R}^d) \subset M_\infty(\mathbb{R}^d). \tag{2.28}$$

$$\|fm\|_q = \|f\|_{1,q} \leq \|f\|_{p,q} . \tag{2.29}$$

2.2.2 Properties

In this section, let us first note that the spaces $L^p_{uloc}(\mathbb{R}^d)$ et $M_\infty(\mathbb{R}^d)$ $(1 \leq p \leq +\infty)$ are special cases of the space $\ell^\infty(\mathbb{R}^d)$.

Definition 2.2.3. Let $1 \leq r \leq \infty$. We have

$$(C_0, \ell^r)(\mathbb{R}^d) = C_0(\mathbb{R}^d) \cap (L^\infty, \ell^r)(\mathbb{R}^d) \tag{2.30}$$

and

$$(L^r, C_0)(\mathbb{R}^d) = \left\{ f \in (L^r, \ell^\infty)(\mathbb{R}^d),\ /\ \|f\|_{L^r(\mathbb{R}^d)} \in C_0(\mathbb{R}^d) \right\}.$$

Theorem 2.2. Let $1 \leq p, q < \infty$. The space $(L^{p'}, \ell^{q'})(\mathbb{R}^d)$ $\left[\text{respectively } (L^{p'}, \ell^1)(\mathbb{R}^d)\right]$ is isometrically isomorphous to$(L^p, \ell^q)'(\mathbb{R}^d)$ $\left[\text{respectively à } (L^p, C_0)'(\mathbb{R}^d)\right]$ through the application

$$g \mapsto \langle f, g \rangle\,, \quad \langle f, g\rangle = \int_{\mathbb{R}^d} f\; g\; dx.$$

For all $g \in (L^{p'}, \ell^{q'})(\mathbb{R}^d)$ $\left[\left(L^{p'}, \ell^1\right)(\mathbb{R}^d)\right]$ and $f \in (L^p, \ell^q)(\mathbb{R}^d)$ $\left[(L^p, C_0)(\mathbb{R}^d)\right]$. we have,

$$|\langle f, g\rangle| \;\leq\; \|f\|_{p,q}\, \|f\|_{p',q'} \quad (1 < p,\ q < +\infty). \tag{2.31}$$

$$|\langle f, g\rangle| \;\leq\; \|f\|_{p,1}\, \|f\|_{p',\infty} \quad (1 \leq p < +\infty). \tag{2.32}$$

Proof. The proof of this theorem is similar to that of Proposition 2.6. See ([21, § 3 P.33]). ■

Proposition 2.8. *The amalgam space* $(L^p, \ell^q)(\mathbb{R}^d)$ $(1 < p,\ q < \infty)$ *is a reflective Banach space.*

Proof. ([21], corollaire 3.3) ■

Theorem 2.3. We have the following statements

(i) $C_c(\mathbb{R}^d)$ is dense in $(C_0, \ell^q)(\mathbb{R}^d)$ pour $1 \leq q \leq \infty$.

(ii) $C_c(\mathbb{R}^d)$ is dense in $(L^p, \ell^q)(\mathbb{R}^d)$ pour $1 \leq p,\ q < \infty$.

(iii) $C_c(\mathbb{R}^d)$ is dense in $(L^p, C_0)(\mathbb{R}^d)$ pour $1 \leq p < \infty$.

Proof. (i) First, note that $C_c(\mathbb{R}^d)$ is included in any amalgam space. Let f be a function in the closure of $C_c(\mathbb{R}^d)$ in $(L^p, \ell^\infty)(\mathbb{R}^d)$. There exists a sequence $(\phi_n) \subset C_c(\mathbb{R}^d)$ such that $\lim\limits_{n\to+\infty} \|\phi_n - f\|_{\infty,q} = 0$. Let $\epsilon > 0$, $\exists n_0, n \in \mathbb{N}$ such that $n > n_0$;

$$\|\phi_n - f\|^q_{\infty,q} = \sum_{k\in\mathbb{Z}^d} \sup_{r>0} \left|\phi_n \chi_{I^r_x} - f\chi_{I^r_x}\right|^q < \epsilon.$$

Since $|\phi_n - f| \leq \|\phi - f\|_{\infty,q}$, ϕ converges uniformly to f on $\mathbb{R}^d$. Therefore, f is continuous and by the definition 2.2.3, $f \in (C_0, \ell^q)(\mathbb{R}^d)$ si q is finite and if q is infinite, then $(C_0, \ell^q)(\mathbb{R}^d) = C_0(\mathbb{R}^d)$ and by density of $C_c(\mathbb{R}^d)$ in $C_0(\mathbb{R}^d)$.

(ii) Let $f \in (L^p, \ell^q)(\mathbb{R}^d)$, as L^p_{loc} is a sub-space dense of (L^p, ℓ^q) for $1 \leq p \leq \infty$, $1 \leq q < \infty$, then $\forall \epsilon > 0$, there exists $g \in L^p_{loc}(\mathbb{R}^d)$ such that $\|f - g\|_{p,q} < \frac{\epsilon}{2}$. If E is the compact support of g then there exists a function h in $C_c(E)$ such that $\|g - h\|_p < \frac{\epsilon}{2}\,|S(E)|^{\frac{1}{q}}$ (car $C_c(E)$ dense in $L^p(E)$). Next, we have $\|g - h\|_{p,q} \leq |S(E)|^{\frac{1}{q}}\, \|g - h\|_p$. Therefore $\|g - h\|_{p,q} < \frac{\epsilon}{2}$. This implies $\|f - h\|_{p,q} \leq \|f - g\|_{p,q} + \|g - h\|_{p,q} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Therefore C_c is dense in (L^p, l^q) for $1 \leq p, q < \infty$.

(iii) Similarly, since $L^p_{loc}(\mathbb{R}^d)$ is dense in $(L^p, C_0)(\mathbb{R}^d)$ and the density of $C_c(\Omega)(\mathbb{R}^d)$ in $L^p(\Omega)(\mathbb{R}^d)$. We have the result.

■

Corollary 2.4. *(i)* $(C_0, \ell^r)(\mathbb{R}^d)$ *is dense in* $(L^p, \ell^q)(\mathbb{R}^d)$ *for* $1 \leq p < \infty,\ 1 \leq r \leq q < \infty$.

(ii) $(L^r, \ell^s)(\mathbb{R}^d)$ *is dense in* $(L^p, \ell^q)(\mathbb{R}^d)$ *for* $1 \leq p \leq r < \infty,\ 1 \leq s \leq q < \infty$.

(iii) $(L^\infty, \ell^s)(\mathbb{R}^d)$ *is dense in* $(L^\infty, \ell^q)(\mathbb{R}^d)$ *for* $1 \leq s < q < \infty$.

(iv) $(L^r, C_0)(\mathbb{R}^d)$ *is dense in* $(L^p, C_0)(\mathbb{R}^d)$ *for* $1 \leq p < r < \infty$.

Proof. This is a direct consequence of Corollary 2.2 and the theorem 2.3. ■

Remark 2.2.3. $C_0(\mathbb{R}^d)$ is a subspace dense of $(L^p, C_0)(\mathbb{R}^d)$, for $1 \leq p \leq \infty$.

Definition 2.2.4. Let A be one of the following spaces $L^p_{uloc}(\mathbb{R}^d)$, $(C_0, \ell^\infty)(\mathbb{R}^d)$ or $(L^p, C_0)(\mathbb{R}^d)$, $1 \leq p \leq \infty$. For all $t \in \mathbb{R}^d$, τ_t denotes the translation operator on A or on $M_s(\mathbb{R}^d)$, $1 \leq s \leq \infty$ define by:

(a) $\tau_t f(s) = f(s-t), f \in A, s \in \mathbb{R}^d$.

(b) $\tau_t \mu(B) = \mu(-t+B), \mu \in M_s(\mathbb{R}^d),\ B \subset \mathbb{R}^d$.

The following theorem shows that for any $t \in \mathbb{R}^d$, τ_t is a bounded operator.

Theorem 2.4. Let $1 \leq p \leq \infty$ and $t \in \mathbb{R}^d$. There exists a constant $c > 0$ such that

(i) $\|\tau_t f\|_{p,\infty} \leq c\,\|f\|_{p,\infty}$, for all $f \in L^p_{uloc}(\mathbb{R}^d)$

(ii) $\|\tau_t \mu\|_\infty \leq c\,\|\mu\|_\infty$, for all $\mu \in M_\infty(\mathbb{R}^d)$.

Proof. Let $t, u \in \mathbb{R}^d$.

let us set $S(t+u) = \left\{t, u \in \mathbb{R}^d /\ I_t \cap I_u \neq \phi\right\}$

$$\begin{aligned}
\|\tau_t f\|_{p,\infty} &= \sup_{x\in\mathbb{R}^d} \|\tau_t f \chi_{I_x}\|_p \\
&= \sup_{x\in\mathbb{R}^d} \left\|f \chi_{I_{x-t}}\right\|_p \\
&= \sup_{(t+u)\in\mathbb{R}^d} \|f \chi_{I_u}\|_p \\
\|\tau_t f\|_{p,\infty} &\leq \sum_{S(t+u)} \sup_{u\in\mathbb{R}^d} \|f \chi_{I_u}\|_p \\
&\leq c\,\|f\|_{p,\infty}.
\end{aligned}$$

(ii) Since $|\tau_t\mu|(\chi_{I_x}) = |\mu|(\chi_{I_{-t+x}}) \leq \sum\limits_{S(-t+x)} |\mu|(\chi_{I_x})$. Therefore,

$$\|\tau_t\mu\|_\infty = \sup_{x\in\mathbb{R}^d} |\tau_t\mu|(\chi_{I_x}) \leq c \sup_{x\in\mathbb{R}^d} |\mu|(\chi_{I_x}) = c\,\|\mu\|_\infty .$$

■

Corollary 2.5. *Let $t \in \mathbb{R}^d$. The translation τ_t is a continuous endomorphism of $L^p_{uloc}(\mathbb{R}^d)$.*

Proof. It follows directly from the theorem 2.4. ■

Lemma 2.1. *(a) if $f \in C_0(\mathbb{R}^d)$, then*

$$\lim_{t\to 0} \|\tau_t f - f\|_\infty = 0.$$

(b) if $f \in L^p(\mathbb{R}^d)$ $(1 \leq p < \infty)$, then

$$\lim_{t\to 0} \|\tau_t f - f\|_p = 0.$$

Proof. (i) Let $f \in C_0(\mathbb{R}^d)$. For all $\epsilon > 0$, there exists a neighbourhood V of 0 such that $|f(x) - f(y)| < \epsilon, \forall y - x \in V$. Next, for every $t \in V$ and $x \in \mathbb{R}^d$, we have :

$$|f(x-t) - f(x)| = |\tau_t f(x) - f(x)| < \epsilon \quad \text{car } \ t = x - (x-t).$$

Since ϵ is arbitrary and V does not depend on x, then $\|\tau_t f - f\|_\infty < \epsilon, \forall t \in V$.

(ii) Let $1 \leq p < \infty$, $f \in L^p(\mathbb{R}^d)$ and $f_1 \in C_c(\mathbb{R}^d)$ such that $\|f - f_1\|_{L^p(\mathbb{R}^d)}$. We have:

$$\begin{aligned}
\|\tau_h f - f\|_{L^p(\mathbb{R}^d)} &\leq \|\tau_h f - \tau_h f_1\|_{L^p(\mathbb{R}^d)} + \|\tau_h f_1 - f_1\|_{L^p(\mathbb{R}^d)} + \|f_1 - f\|_{L^p(\mathbb{R}^d)} \\
&\leq \epsilon + \epsilon + \|\tau_h f_1 - f_1\|_{L^p(\mathbb{R}^d)} .
\end{aligned}$$

On the other hand, it comes that $\lim\limits_{h\to 0} \|\tau_h f_1 - f_1\|_{L^p(\mathbb{R}^d)} = 0$, thus $\lim\limits_{h\to 0} \|\tau_h f - f\|_{L^p(\mathbb{R}^d)} = 0$.

■

Theorem 2.5. Let $1 \leq p, q < \infty$. If $f \in (L^p, \ell^q)(\mathbb{R}^d)$, $(L^p, C_0)(\mathbb{R}^d)$ ou $(C_0, \ell^s)(\mathbb{R}^d)$, $1 \leq s \leq \infty$, then the application $t \mapsto \tau_t f$ is continuous on $\mathbb{R}^d$.

Proof. Suppose initially that $f \in (C_0, \ell^q)(\mathbb{R}^d)$, by virtue of the theorem 2.3, there exists $g \in C_c(\mathbb{R}^d)$ such that $\|f - g\|_{\infty,q} < \epsilon$. Therefore, according to Lemma 2.2, we have $\|\tau_t g - g\|_\infty < \epsilon$. Therefore,

$$\begin{aligned}
\|\tau_t f - f\|_{\infty,q} &\leq \|\tau_t f - \tau_t g\|_{\infty,q} + \|\tau_t g - g\|_{\infty,q} + \|g - f\|_{\infty,q} \\
&\leq C\,\|f - g\|_{\infty,q} + \epsilon + \epsilon.
\end{aligned}$$

From $\lim_{t\to 0} \|\tau_t f - f\|_{\infty,q} = 0.$

The proof for $f \in (L^p, C_0)(\mathbb{R}^d)$ is similar. Let $f \in (L^p, \ell^q)(\mathbb{R}^d)$, par densité de $(C_0, \ell^q)(\mathbb{R}^d)$ dans $(L^p, \ell^q)(\mathbb{R}^d)$, $(1 \leq p,\ q < \infty)$, there exists $g \in (C_0, \ell^q)(\mathbb{R}^d)$ such that $\|f - g\|_{p,q}$. Therefore,

$$\begin{aligned}
\|\tau_t f - f\|_{p,q} &\leq \|\tau_t f - \tau_t g\|_{p,q} + \|\tau_t g - g\|_{p,q} + \|f - g\|_{p,q} \\
&\leq C\ \|f - g\|_{p,q} + \|\tau_t g - g\|_{\infty,q} + \|g - f\|_{p,q} \\
&\leq (C+1)\epsilon + \|\tau_t g - g\|_{\infty,q}.
\end{aligned}$$

Since ϵ does not depend of t. It comes that

$$\|\tau_t f - f\|_{p,q} \leq \|\tau_t g - g\|_{\infty,q} < \epsilon.$$

Thus

$$\lim_{t\to 0} \|\tau_t f - f\|_{p,q} = 0.$$

The case $q = \infty$ leads from the lemma 2.1. ■

2.2.3 Usual and convolution product

In this section, we introduce two operations on amalgam spaces and unbounded measure spaces of type q: the usual and convolution product. These operations have been previously studied [3], our first result is a generalization of the product in L^p-spaces.

Proposition 2.9. *Let $1 \leq p, q, r, s \leq \infty$ such that $\frac{1}{p} + \frac{1}{r} = \frac{1}{m} \leq 1$ and $\frac{1}{q} + \frac{1}{s} = \frac{1}{n} \leq 1$ then, we have:*

(a) $(L^p, \ell^q)(\mathbb{R}^d)\,(L^r, \ell^s)(\mathbb{R}^d) \subseteq (L^m, \ell^n)(\mathbb{R}^d)$,

(b) $(C_0, \ell^q)(\mathbb{R}^d)\,(C_0, \ell^s)(\mathbb{R}^d) \subseteq (C_0, \ell^n)(\mathbb{R}^d)$,

(c) $(L^p, C_0)(\mathbb{R}^d)\,(L^r, C_0)(\mathbb{R}^d) \subseteq (L^m, C_0)(\mathbb{R}^d)$.

Furthermore, if $f \in (L^p, \ell^q)(\mathbb{R}^d)$ and $g \in (L^r, \ell^s)(\mathbb{R}^d)$, then we have:

$$\|fg\|_{m,n} \leq \|f\|_{p,q}\,\|g\|_{r,s}.$$

Proof. Soient $1 \leq p, q, r, s < \infty$, $f \in (L^p, \ell^q)(\mathbb{R}^d)$ et $g \in (L^r, \ell^s)(\mathbb{R}^d)$. We will apply Holder's inequality twice.

$$\int_{\mathbb{R}^d} |fg|^m\,dx = \int_{\mathbb{R}^d} (|f|^p)^{\frac{m}{p}}\,(|g|^r)^{\frac{m}{r}}\,dx \leq \left(\int_{\mathbb{R}^d} |f|^p\,dx\right)^{\frac{m}{p}} \left(\int_{\mathbb{R}^d} |g|^r\,dx\right)^{\frac{m}{r}}.$$

Therefore, applying Holder's inequality a second time, we have:

$$\begin{aligned}
\|fg\|^n_{m,n} &= \sum_{k\in\mathbb{Z}^d}\left(\int |fg|^m\,dx\right)^{\frac{n}{m}}\\
&\leq \sum_{k\in\mathbb{Z}^d}\left(\int |f|^p\,dx\right)^{\frac{n}{p}}\left(\int |g|^r\,dx\right)^{\frac{n}{r}}\\
&\leq \sum_{k\in\mathbb{Z}^d}\left[\left(\int |f|^p\,dx\right)^{\frac{q}{p}}\right]^{\frac{n}{q}}\left[\left(\int |g|^r\,dx\right)^{\frac{s}{r}}\right]^{\frac{n}{s}}\\
&\leq \left[\sum_{k\in\mathbb{Z}^d}\|f\|^q_p\right]^{\frac{n}{q}}\left[\sum_{k\in\mathbb{Z}^d}\|g\|^s_r\right]^{\frac{n}{s}}\\
&\leq \|f\|_{p,q}\,\|g\|_{r,s}\,.
\end{aligned}$$

From (a).

(b) follows from (a) and the definition 2.2.3 (the case $q = s = infty$ is well known).

Now, let $f \in (L^p, C_0)(\mathbb{R}^d)$ et $g \in (L^r, C_0)(\mathbb{R}^d)$, according to (a), $fg \in (L^m, \ell^\infty)(\mathbb{R}^d)$ et

$$\|fg\|_m \leq \|f\|_p\,\|g\|_r\,.$$

This implies that $\lim\|fg\|_m \leq \lim\|f\|_p\,\|g\|_r = 0$, (car $\|f\|_p$ and $\|g\|_r$ are continues). Hence, $fg \in (L^m, C_0)(\mathbb{R}^d)$. From (c). ■

Proposition 2.10. *Let* $1 \leq q, s \leq \infty$. *Then* $M_q(\mathbb{R}^d) \times M_s(\mathbb{R}^d) \subseteq M_n(\mathbb{R}^d \times \mathbb{R}^d)$, *where* $n = \max(q, s)$ *and we have:*

$$\|\mu \times v\|_n \leq \|\mu\|_q\,\|v\|_s \quad \textit{pour tout}\ \ \mu \in M_q(\mathbb{R}^d),\ v \in M_s(\mathbb{R}^d).$$

(2.33)

Proof. Let $\mu \in M_q(\mathbb{R}^d)$ and $v \in M_s(\mathbb{R}^d)$, we have:

$$\begin{aligned}
\|\mu \times v\|^n_n &= \sum_{k\in\mathbb{Z}^d}\|\mu \times v\|\,(\chi_{J_k})^n\\
&= \sum_{k\in\mathbb{Z}^d}|\mu|\,|v|\,(\chi_{J_k})^n\\
&\leq \sum_{k\in\mathbb{Z}^d}|\mu|\,(\chi_{J_k})^n\,|v|\,(\chi_{J_k})^n\\
&\leq \|\mu\|^n_n\,\|v\|^n_n\\
&\leq \|\mu\|^n_q\,\|v\|^n_s\,.
\end{aligned}$$

■

Theorem 2.6. [21] Let $1 \leq q, s \leq \infty$ such that $\frac{1}{q} + \frac{1}{s} - 1 = \frac{1}{n} \leq 1$. There exists a constant $C > 0$ such that for all $\mu \in M_q(\mathbb{R}^d)$ and $v \in M_s(\mathbb{R}^d)$, μ and v convolvable and $\mu * v \in M_n(\mathbb{R}^d)$. Moreover,

$$\|\mu * v\|_n \leq C \ \|\mu\|_q \|v\|_s .$$

Corollary 2.6. *If $f \in (L^1, \ell^q)(\mathbb{R}^d)$, $1 \leq q \leq \infty$, $\mu \in M_s(\mathbb{R}^d)$, $1 \leq s \leq \infty$ and $\frac{1}{q}+\frac{1}{s}-1 = \frac{1}{n} \leq 1$, then f as a measure convolves with μ, $f * \mu$ is an absolutely continuous measure and we have*

$$(f * \mu)(y) = \int_{\mathbb{R}^d} f(y - x) \, d\mu(x).$$

Moreover,

$$f * \mu \in \left(L^1, \ell^n\right)(\mathbb{R}^d) \quad \textit{and} \quad \|f * \mu\|_{1,n} \leq C \ \|f\|_{1,q} \|\mu\|_s .$$

Proof. Let $f \in (L^1, \ell^n)(\mathbb{R}^d) \subseteq M_q(\mathbb{R}^d)$, $1 \leq q \leq \infty$, then according to the theorem 2.6, $f * \mu \in M_n(\mathbb{R}^d)$. In other hand,

$$\|f * \mu\|_{1,n} = \|(f * \mu)_m\|_n \leq \|f\|_{1,q} \|\mu\|_s .$$

■

Corollary 2.7. *Let $1 \leq q, s \leq \infty$. If $f \in (L^1, \ell^q)(\mathbb{R}^d)$, $g \in (L^1, \ell^s)(\mathbb{R}^d)$ and $\frac{1}{p} + \frac{1}{s} = \frac{1}{n} \leq 1$, then $f * g \in (L^1, \ell^n)(\mathbb{R}^d)$ and*

$$f * g(y) = \int_{\mathbb{R}^d} f(y - x)g(x)dx.$$

Furthermore,

$$\|f * g\|_{1,n} \leq \|f\|_{1,q} \|g\|_{1,s} .$$

Proof. According to the identities (2.27), (2.29) and the corollary 2.6, we have: $f*g \in (L^1, \ell^n)(\mathbb{R}^d)$ and

$$\|f * g\|_{1,n} = \|(f * g)_m\|_n \leq \|f\|_{1,q} \|g\|_{1,s} .$$

■

Theorem 2.7. Let $1 \leq p, q, r, s \leq \infty$ such that $\frac{1}{p} + \frac{1}{r} - 1 = \frac{1}{m} \leq 1$ and $\frac{1}{q} + \frac{1}{s} - 1 = \frac{1}{n} \leq 1$, then:

(i) $(L^p, \ell^q) * (L^r, \ell^s) \subseteq (L^m, \ell^n)$;

(ii) $(L^p, \ell^q) * \left(L^{p'}, \ell^{q'}\right) \subseteq C_0, \qquad 1 \leq p \leq \infty, \ 1 < q < \infty;$

(iii) $(L^p, \ell^q) * \left(L^{p'}, \ell^{s}\right) \subseteq (C_0, \ell^n), \qquad 1 \leq p, s \leq \infty, \ 1 < q < \infty.$

(iv) $(L^p, \ell^q) * \left(L^{r}, \ell^{q'}\right) \subseteq (L^m, C_0), \qquad 1 \leq p, r \leq \infty, \ 1 < q < \infty.$

Furthermore, if $f \in (L^p, \ell^q)(\mathbb{R}^d)$ and $g \in f \in (L^r, \ell^s)(\mathbb{R}^d)$, then $f * g \in f \in (L^m, \ell^n)(\mathbb{R}^d)$ and we have:

(A.1) $\|f * g\|_{m,n} \leq C \ \|f\|_{p,q} \|g\|_{r,s}$ if $m \neq 1$ and $C > 0$;

(A.2) $\|f * g\|_{1,n} \leq C \ \|f\|_{1,q} \|g\|_{1,s}$.

Proof. For $p = r = 1$, then (i) and (A.2) follow from the corollary 2.7. Next, for $1 < p, q, r, s < \infty$, we set:

(1) $\frac{p}{m} + \frac{p}{r'} = 1,\ \frac{r}{m} + \frac{r}{p'} = 1,\ \frac{m'}{p'} + \frac{m'}{r'} = 1.$

(2) $\frac{q}{n} + \frac{q}{s'} = 1,\ \frac{s}{n} + \frac{s}{q'} = 1,\ \frac{n'}{q'} + \frac{n'}{s'} = 1.$

Let $\phi \in C_c(\mathbb{R}^d)$ and $x, y \in \mathbb{R}^d$, we have from (1) that:

$$\int_{J_k^r}\int_{J_k^r} |\phi(x+y)| \, |f(x)| \, |g(y)| \, dxdy = \int_{J_k^r}\int_{J_k^r} \left(|f(x)|^p \, |g(x)|^r\right)^{\frac{1}{m}} \left(|\phi(x+y)|^{m'} \, |f(x)|^p\right)^{\frac{1}{r'}} \left(|\phi(x+y)|^{m'} \, |g(x)|^r\right)^{\frac{1}{p'}} dxdy.$$

Using Holder's inequality for : $\alpha_1 = \frac{1}{m}$, $\alpha_2 = \frac{1}{r'}$, $\alpha_3 = \frac{1}{p'}$, $f_1(x,y) = |f(x)|^p \, |g(x)|^r$, $f_2(x,y) = \phi(x+y)^{m'} \, |f(x)|^p$, $f_3(x,y) = |\phi(x+y)|^{m'} \, |g(x)|^r$ we have,

$$\int_{J_k^r}\int_{J_k^r} |\phi(x+y)| \, |f(x)| \, |g(y)| \, dxdy \leq \left[\int_{J_k^r}\int_{J_k^r} f_1(x,y)dxdy\right]^{\alpha_1} \left[\int_{J_k^r}\int_{J_k^r} f_2(x,y)dxdy\right]^{\alpha_2} \left[\int_{J_k^r}\int_{J_k^r} f_3(x,y)dxdy\right]^{\alpha_3}.$$

Or

$$\int_{J_k^r}\int_{J_k^r} f_1(x,y)dxdy = \left(\int_{J_k^r} |f(x)|^p \, dx\right)\left(\int_{J_k^r} |g(x)|^r \, dy\right) = \|f\|^p_{L^p(J_k^r)} \|g\|^r_{L^r(J_k^r)}.$$

De même, on a:

$$\begin{aligned}
\int_{J_k^r}\int_{J_k^r} f_2(x,y)dxdy &= \int_{J_k^r}\int_{J_k^r} |\phi(x+y)|^{m'} \, |f(x)|^p \, dxdy \\
&= \int_{{}_k^r} |f(x)|^p \int_{J_k^r} |\phi(-x-y)|^{m'} \, dxdy, \quad \check{\phi}(x) = \phi(-x) \\
&= \int_{J_k^r} |f(x)|^p \int_{J_k^r} \left|\tau_x\check{\phi}(y)\right|^{m'} dxdy \\
\int_{J_k^r}\int_{J_k^r} f_2(x,y)dxdy &\leq C \ \|\phi\|^{m'}_{L^{m'}(J_k^r)} \|f\|^p_{L^p(J_k^r)}.
\end{aligned}$$

In the same way+,

$$\int_{J_k^r}\int_{J_k^r} f_2(x,y)dxdy \leq C\ \|\phi\|^{m'}_{L^{m'}(J_k^r)} \|g\|^r_{L^r(J_k^r)}.$$

From (1) and (2), we get:

$$\begin{aligned}\int_{J_k^r}\int_{J_k^r} |\phi(x+y)|\,|f(x)|\,|g(y)|\,dxdy \leq\ & C\ \left[\|f\|^p_{L^p(J_k^r)}\|g\|^r_{L^r(J_k^r)}\right]^{\frac{1}{m}}\left[\|\phi\|^{m'}_{L^{m'}(J_k^r)}\|f\|^p_{L^p(J_k^r)}\right]^{\frac{1}{r'}} \\ & \left[\|\phi\|^{m'}_{L^{m'}(J_k^r)}\|g\|^r_r\right]^{\frac{1}{p'}} \\ =\ & C\ \|\phi\|_{L^{m'}(J_k^r)}\|f\|_{L^p(J_k^r)}\|g\|_{L^r(J_k^r)} \\ =\ & C\ \left[\|f\|^q_{L^p(J_k^r)}\|g\|^s_{L^r(J_k^r)}\right]^{\frac{1}{n}}\left[\|\phi\|^{n'}_{L^{m'}(J_k^r)}\|f\|^n_{L^p(J_k^r)}\right]^{\frac{1}{s'}} \\ & \left[\|\phi\|^{n'}_{L^{m'}(J_k^r)}\|g\|^s_{L^r(J_k^r)}\right]^{\frac{1}{q'}}.\end{aligned}$$

Let's apply Holder's inequality a second time for $\alpha_1 = \frac{1}{n},\ \alpha_2 = \frac{1}{s'},\ \alpha_3 = \frac{1}{p'},\ f_1 = \|f\|^q_{L^p(J_k^r)}\|g\|^s_{L^r(J_k^r)}$, $f_2 = \|\phi\|^{n'}_{L^{m'}(J_k^r)}\|f\|^n_{L^p(J_k^r)},\ f_3 = \|\phi\|^{n'}_{L^{m'}(J_k^r)}\|g\|^s_{L^r(J_k^r)}$. We get:

$$\sum_{k\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}\int_{J_k^r}\int_{J_k^r}|\phi(x+y)|\,|f(x)|\,|g(y)|\,dxdy \leq C\left[\sum_{k\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d} f_1\right]^{\alpha_1}\left[\sum_{k\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d} f_2\right]^{\alpha_2}\left[\sum_{k\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d} f_3\right]^{\alpha_3}.$$

Or

$$\sum_{k\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d} f_1 \leq \sum_{k\in\mathbb{Z}^d}\|f\|^q_p\sum_{k\in\mathbb{Z}^d}\|g\|^s_r \leq \|f\|^q_{p,q}\|g\|^s_{r,s}.$$

In the same way, $\sum\limits_{k\in\mathbb{Z}^d}\sum\limits_{k\in\mathbb{Z}^d} f_2 \leq \|\phi\|^{n'}_{m',n'}\|f\|^q_{p,q}$ and $\sum\limits_{k\in\mathbb{Z}^d}\sum\limits_{k\in\mathbb{Z}^d} f_3 \leq \|\phi\|^{n'}_{m',n'}\|g\|^s_{r,s}$. Thus, we conclude that:

$$\begin{aligned}\int_{J_k^r}\int_{J_k^r}|\phi(x+y)|\,|f(x)|\,|g(y)|\,dxdy\ &\leq\ \sum_{k\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}\int_{J_k^r}\int_{J_k^r}|\phi(x+y)|\,|f(x)|\,|g(y)|\,dxdy \\ &\leq\ C\ \left[\|f\|^q_{p,q}\|g\|^s_{r,s}\right]^{\frac{1}{n}}\left[\|\phi\|^{n'}_{m',n'}\|f\|^q_{p,q}\right]^{\frac{1}{s'}}\left[\|\phi\|^{n'}_{m',n'}\|g\|^s_{r,s}\right]^{\frac{1}{q'}} \\ &=\ C\ \|\phi\|_{m',n'}\|f\|_{p,q}\|g\|_{r,s}.\end{aligned}$$

As a result of the above, the linear functional defined by:

$$T(\phi) = \int_{\mathbb{R}^d}\phi(x)\ f*g(x)dx, \quad \phi\in C_c(\mathbb{R}^d)$$

holds $\|T(\phi)\| \leq C\ \|\phi\|_{m',n'}\|f\|_{p,q}\|g\|_{r,s}$. Since $C_c(\mathbb{R}^d)$ is dense in $\left(L^{m'}, l^{n'}\right)(\mathbb{R}^d)$ [respectively in $\left(L^{m'}, C_0\right)(\mathbb{R}^d)$ if $(n=1)$], T extends uniquely into a continuous linear form which is again denoted by T in $(L^{m'}, l^{n'})'(\mathbb{R}^d) = (L^m, l^n)(\mathbb{R}^d)$ [respectively in $(L^{m'}, C_0)'(\mathbb{R}^d) = (L^m, l^1)(\mathbb{R}^d)$] such that $\|T\| \leq C\ \|f\|_{p,q}\|g\|_{r,s}$ according to the theorem 2.2. Therefore, we have:

$$\|f*g\|_{m,n} \leq C\ \|f\|_{p,q}\|g\|_{r,s}.$$

which proves (i) and (A.1).

Let $f \in (L^p, \ell^q)(\mathbb{R}^d)$ and $g \in (L^{p'}, \ell^{q'})(\mathbb{R}^d)$. From (i), $f * g \in L^\infty(\mathbb{R}^d)$. Let us assume that $1 \leq p < \infty$, we have for all $t,\ s \in \mathbb{R}^d$:

$$f * g(t) = \int_{\mathbb{R}^d} f(t-x)g(x)dx = \int_{\mathbb{R}^d} \tau_t f(x) g(x) dx = \langle \tau_t f, g\rangle .$$

according to the theorem 2.2, on a:

$$\begin{aligned} |f * g(t) - f * g(s)| &= |\langle \tau_t f, g\rangle - \langle \tau_s f, g\rangle| \\ &= |\langle \tau_t f - \tau_s f, g\rangle| \\ &\leq \|g\|_{p',q'} \|\tau_t f - \tau_s f\|_{p,q} . \end{aligned}$$

Since $\|\tau_t f - \tau_s f\|_{p,q} \to 0$, according to the theorem 2.5, then we conclude that $f * g$ is continuous. Now, since $C_c(\mathbb{R}^d)$ est dense dans $(L^p, \ell^q)(\mathbb{R}^d)$ et $L^{p'}_{loc}(\mathbb{R}^d)$ is dense in $\left(L^{p'}, \ell^{q'}\right)(\mathbb{R}^d)$, let $\epsilon > 0$, there exists $\phi \in C_c(\mathbb{R}^d)$ et $h \in L^{p'}_{loc}(\mathbb{R}^d)$ such that:

$$\|\phi - f\|_{p,q} < \frac{\epsilon}{\|g\|_{p',q'}} \quad \text{et} \quad \|h - g\|_{p',q'} < \frac{\epsilon}{\|\phi\|_{p,q}}.$$

This implies by $(A.1)$ that:

$$\begin{aligned} \|\phi * h - f * g\|_\infty &\leq \|\phi * h - \phi * g\|_\infty + \|\phi * g - f * g\|_\infty \\ &\leq \|\phi\|_{p,q} \|h - g\|_{p',q'} + \|g\|_{p',q'} \|\phi - f\|_{p,q} \\ &< \epsilon. \end{aligned}$$

Since ϵ is arbitrary and $\phi * h \in C_c(\mathbb{R}^d)$, this means that $f * g$ is the closure of $C_c(\mathbb{R}^d)$ in the space of continuous functions on $\mathbb{R}^d$, given $C_c(\mathbb{R}^d)$ is dense in $C_0(\mathbb{R}^d)$, then $f * g \in C_0(\mathbb{R}^d)$.

If $p = +\infty$, so we reason the same way.

Or now $f \in (L^p, \ell^q)(\mathbb{R}^d)$ and $g \in \left(L^{p'}, \ell^s\right)(\mathbb{R}^d)$. From (i), $f * g \in (L^\infty, \ell^n)(\mathbb{R}^d)$. Since $\frac{1}{s} = \frac{1}{n} + \frac{1}{q'} \geq \frac{1}{q'}$, alors $\left(L^{p'}, \ell^s\right)(\mathbb{R}^d) \subseteq \left(L^{p'}, \ell^{q'}\right)(\mathbb{R}^d)$, par (ii), $f * g \in C_0(\mathbb{R}^d)$. Hence (iii).

Let $f \in (L^p, \ell^q)(\mathbb{R}^d)$ and $g \in \left(L^r, \ell^{q'}\right)(\mathbb{R}^d)$. Since $L^p_c(\mathbb{R}^d)$ and $L^p_c(\mathbb{R}^d)$ are dense in $(L^p, \ell^q)(\mathbb{R}^d)$ et $(L^r, \ell^s)(\mathbb{R}^d)$ respectively, then for all $\epsilon > 0$, there exists $\phi \in L^p_c(\mathbb{R}^d)$ and $\psi \in L^r_c(\mathbb{R}^d)$ such that:

$$\|f - \phi\|_{p,q} < \frac{\epsilon}{\|g\|_{r,q'}} \quad \text{et} \quad \|g - \psi\|_{r,s} < \frac{\epsilon}{\|\phi\|_{p,q}}.$$

From (A.1), we have:

$$\begin{aligned} \|f * g - \phi * \psi\|_{m,\infty} &\leq \|f * g - \phi * \psi\|_{m,\infty} + \|\phi * g - \phi * \psi\|_{m,\infty} \\ &\leq \|g\|_{r,q'} \|f - \phi\|_{p,q} + \|\phi\|_{p,q} \|g - \phi\|_{r,q'} \\ &< \epsilon. \end{aligned}$$

Since ϵ is arbitrary and $\phi * \psi \in L^m_c(\mathbb{R}^d)$, then $f * g$ is the closure of $L^m_c(\mathbb{R}^d)$ in $(L^m, \ell^\infty)(\mathbb{R}^d)$ because according to (i), $f * g \in (L^m, \ell^\infty)(\mathbb{R}^d)$. By density of $L^m_c(\mathbb{R}^d)$ in $(L^m, C_0)(\mathbb{R}^d)$, we conclude that $f * g \in (L^m, C_0)(\mathbb{R}^d)$. Hence (iv). ■

Remark 2.2.4. Identities (A.1) and (A.2) are **Young's inequality** for amalgam.

Theorem 2.8. Let $1 \le p, q, s \le \infty$. If $\frac{1}{q} + \frac{1}{s} - 1 = \frac{1}{n} \le 1$, then,

(i) $(L^p, \ell^q) * M_s \subseteq (L^p, \ell^n)$,

(ii) $(L^p, \ell^q) * M_{q'} \subseteq (L^p, C_0), \;\; 1 \le p \le \infty, 1 < q < \infty$,

(iii) $(C_0, \ell^q) * M_s \subseteq (C_0, \ell^n), \;\; 1 \le q \le \infty$.

Hence, $(C_0, \ell^n) * M_{q'} \subseteq C_0, \;\; 1 \le q \le \infty$. Furthermore, if $f \in (L^p, \ell^q)(\mathbb{R}^d)$ et $\mu \in M_s(\mathbb{R}^d)$, then:

(A.3) $\|f * \mu\|_{p,n} \le \|f\|_{p,q} \|\mu\|_s, \;\;$ si $\; p \neq 1$.

(A.4) $\|f * \mu\| \le \|f\|_{1,n} \|\mu\|_s$.

Proof. We reason in a similar way to the proof of the theorem 2.7. ■

2.3 $(L^p_{uloc})^\alpha(\mathbb{R}^d)$ Spaces

We aim to define the space $(L^p_{uloc})^\alpha(\mathbb{R}^d)$ which is a particular case of the amalgam $(L^p, \ell^q)^\alpha(\mathbb{R}^d)$, $1 \le p \le \alpha \le q \le \infty$.

In [7], I. Fofana defines space $(L^p, \ell^q)^\alpha(\mathbb{R}^d)$ as follows:

$$(L^p, \ell^q)^\alpha(\mathbb{R}^d) = \left\{ f \in L_0(\mathbb{R}^d) \; / \; \|f\|_{p,q,\alpha} < +\infty \right\},$$

with

$$\|f\|_{p,q,\alpha} = \sup_{r>0} r^{d\left(\frac{1}{\alpha} - \frac{1}{p} - \frac{1}{q}\right)} {}_r\|f\|_{p,q}, \quad f \in L_0(\mathbb{R}^d). \tag{2.34}$$

Definition 2.3.1. For all $1 \le p \le \alpha \le \infty$, We have:

$$(L^p_{uloc})^\alpha(\mathbb{R}^d) = \left\{ f \in L_0(\mathbb{R}^d) \; / \; \|f\|_{p,\infty,\alpha} < +\infty \right\},$$

with

$$\|f\|_{p,\infty,\alpha} = \sup_{r>0} r^{d\left(\frac{1}{\alpha} - \frac{1}{p}\right)} {}_r\|f\|_{p,\infty}, \quad f \in L_0(\mathbb{R}^d). \tag{2.35}$$

Proposition 2.11. *Let* $1 \le p \le \alpha \le \infty$.

(a) $(L^p_{uloc})^\alpha(\mathbb{R}^d)$ *is a vector subspace of the vector space* $L^p_{uloc}(\mathbb{R}^d)$.

(b) The application $f \mapsto \|f\|_{p,\infty,\alpha}$ defines a norm on $\left(L^p_{uloc}\right)^\alpha(\mathbb{R}^d)$.

Proof. Let $1 \leq p \leq \alpha \leq \infty$.

(a) • $\left(L^p_{uloc}\right)^\alpha(\mathbb{R}^d) \neq \phi$ because it contains 0.

• Moreover, if f and g are two elements of $\left(L^p_{uloc}\right)^\alpha(\mathbb{R}^d)$, et $\beta \in \mathbb{R}$, then we have:

$$\|\beta f\|_{p,\infty,\alpha} = \sup_{r>0} r^{d\left(\frac{1}{\alpha}-\frac{1}{p}\right)} \|\beta f\|_{p,\infty} = |\beta| \sup_{r>0} r^{d\left(\frac{1}{\alpha}-\frac{1}{p}\right)} \|f\|_{p,\infty} = |\beta| \|f\|_{p,\infty,\alpha}. \tag{2.36}$$

Next,

$$\begin{aligned} \|f+g\|_{p,\infty,\alpha} &= \sup_{r>0} r^{d\left(\frac{1}{\alpha}-\frac{1}{p}\right)} \|f+g\|_{p,\infty} \\ &\leq \sup_{r>0} r^{d\left(\frac{1}{\alpha}-\frac{1}{p}\right)} \left(\|f\|_{p,\infty} + \|g\|_{p,\infty}\right) \\ &\leq \|g\|_{p,\infty,\alpha} + \|g\|_{p,\infty,\alpha}. \end{aligned} \tag{2.37}$$

Thus, $\left(L^p_{uloc}\right)^\alpha(\mathbb{R}^d)$ is a sub-vector space of $L^p_{uloc}(\mathbb{R}^d)$.

(b) Let $f \in \left(L^p_{uloc}\right)^\alpha(\mathbb{R}^d)$ such that $\|f\|_{p,\infty,\alpha} = 0$. As defined in the standard $\|.\|_{p,\infty,\alpha}$, we have ${}_r\|f\|_{p,\infty} = 0$ for all $r > 0$. Since $\|.\|_{p,\infty}$ is a norm then $f = 0$.
The inequalities (2.36) and (2.37) allow us to conclude that the application $f \mapsto \|f\|_{p,\infty,\alpha}$ is a norm on $\left(L^p_{uloc}\right)^\alpha(\mathbb{R}^d)$.

■

Proposition 2.12. $\left(L^p_{uloc}\right)^\alpha(\mathbb{R}^d)$, $\|.\|_{p,\infty,\alpha}$ *is a Banach space, for* $1 \leq p \leq \alpha \leq \infty$.

2.3.1 Some subsets of $\left(L^p_{uloc}\right)^\alpha(\mathbb{R}^d)$

Proposition 2.13. *Let* $1 \leq p < +\infty$ *et* $f \in \left(L^p_{uloc}\right)^\alpha(\mathbb{R}^d)$.

$$\lim_{r\to+\infty} {}_r\|f\|_{p,\infty} = \|f\|_p = \sup_{r>0} {}_r\|f\|_{p,\infty}. \tag{2.38}$$

Proof. Let $f \in \left(L^p_{uloc}\right)^\alpha(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ and $r > 0$. We have

$$\left\|f\chi_{I^1_x}\right\|_p = \left[\int_{I^1_x} |f(x)|^p \, d(t)\right]^{\frac{1}{p}} \leq \left[\int_{\mathbb{R}^d} |f(x)|^p \, d(t)\right]^{\frac{1}{p}} = \|f\|_p.$$

So for any real number $r > 0$, we have:

$${}_r\|f\|_{p,\infty} = \sup_{x\in\mathbb{R}^d} \left\|f\chi_{I^r_x}\right\|_p \leq \|f\|_p.$$

Hence,

$$\sup_{r>0} {}_r\|f\|_{p,\infty} = \lim_{r\to+\infty} {}_r\|f\|_{p,\infty} \leq \|f\|_p.$$

- If $\sup\limits_{r>0} {}_r\left\|f\right\|_{p,\infty} = +\infty$, then the equality follows.
- Let assume that $\sup\limits_{r>0} {}_r\left\|f\right\|_{p,\infty} = M < +\infty$. For any real number $r > 0$, we have: ${}_r\left\|f\right\|_{p,\infty} \leq M$. Next: $\left\|f\right\|_{p,\infty} \leq M^p$. Therefore

$$\left\|f\right\|_{p,\infty} \leq M = \sup_{r>0} {}_r\left\|f\right\|_{p,\infty} \quad \text{car} \quad \sup \left\|f\right\|_p = \left\|f\right\|_p^p.$$

■

In the following proposals, we examine the relationship between the spaces $\left(L^p_{uloc}\right)^\alpha(\mathbb{R}^d)$ and those of Lebesgue. We also justify the condition $p \leq \alpha$ that we use in the definition of spaces $L^p_{uloc}(\mathbb{R}^d)$.

Proposition 2.14. *Let* $1 \leq p \leq \alpha \leq \infty$. *Then*

$$\left\|f\right\|_{p,\infty,\alpha} \leq \left\|f\right\|_\alpha, \quad \textit{for all} \ \ f \in L_0(\mathbb{R}^d). \tag{2.39}$$

Proof. Let $f \in L_0(\mathbb{R}^d)$.

- If $p = \alpha$, then, for any real $r > 0$,

$${}_r\left\|f\chi_{I_x^r}\right\|_p \leq \left\|f\right\|_p = \left\|f\right\|_\alpha, \qquad x \in \mathbb{R}^d.$$

Thus,

$$\left\|f\right\|_{p,\infty} = \sup_{r>0} {}_r\left\|f\chi_{I_x^r}\right\|_{p,\infty} \leq \left\|f\right\|_\alpha.$$

- If $p < \alpha \leq +\infty$, then, for each real $r > 0$,

$${}_r\left\|f\right\|_{p,\infty} \leq r^{d\left(\frac{1}{p}-\frac{1}{\alpha}\right)}\left\|f\chi_{I_x^r}\right\|_\alpha \leq r^{d\left(\frac{1}{p}-\frac{1}{\alpha}\right)}\left\|f\right\|_\alpha.$$

Hence,

$$\left\|f\right\|_{p,\infty} = \sup_{x\in\mathbb{R}^d} \left\|f\chi_{I_x^r}\right\|_p \leq r^{d\left(\frac{1}{p}-\frac{1}{\alpha}\right)}\left\|f\right\|_\alpha.$$

Therefore,

$$r^{d\left(\frac{1}{\alpha}-\frac{1}{p}\right)}\left\|f\right\|_{p,\infty} \leq \left\|f\right\|_\alpha.$$

So

$$\left\|f\right\|_{p,\infty,\alpha} = \sup_{r>0} r^{d\left(\frac{1}{\alpha}-\frac{1}{p}\right)}\left\|f\right\|_{p,\infty} \leq \left\|f\right\|_\alpha.$$

■

Remark 2.3.1. The above proposition indicates that $L^\alpha(\mathbb{R}^d) \subset \left(L^p_{uloc}\right)^\alpha(\mathbb{R}^d)$, for $1 \leq p \leq \alpha \leq +\infty$.

The following proposition shows that this inclusion becomes an equality if $\alpha = p$.

Proposition 2.15. *Let* $1 \leq p \leq +\infty$. *For any element* f *of* $L_0(\mathbb{R}^d)$, *there exists a constant* $C > 0$, *such that:*

$$C \left\| f \right\|_p \leq \left\| f \right\|_{p,\infty,\alpha} \leq \left\| f \right\|_p .$$

Proof. Let $f \in L_0(\mathbb{R}^d)$, according to the proposition 2.14, on a:

$$\left\| f \right\|_{p,\infty,\alpha} \leq \left\| f \right\|_p .$$

Since there is equivalence between the norms ${}_1\left\|.\right\|_{p,\infty}$ et ${}_r\left\|.\right\|_{p,\infty}$, there exists a constant $C > 0$ such that:

$${}_r\left\| f \right\|_{p,\infty} \leq \left\| f \right\|_{p,\infty} .$$

Thus,

$$\sup_{r>0} {}_r\left\| f \right\|_{p,\infty} \leq C \sup_{r>0} \left\| f \right\|_{p,\infty} = C \left\| f \right\|_{p,\infty,p} .$$

However, according to the proposition 2.13,

$$\left\| f \right\|_p = \sup_{r>0} {}_r\left\| f \right\|_{p,\infty} .$$

D'où,

$$\frac{1}{C} \left\| f \right\|_p \leq \left\| f \right\|_{p,\infty,\alpha} \leq \left\| f \right\|_p .$$

■

By defining the spaces $L^p_{uloc}(\mathbb{R}^d)$, we always set the condition $p \leq \alpha \leq +\infty$. One is entitled to wonder what would happen if this condition were not met. $\alpha > +\infty$, then $\left\| f \right\|_{p,\infty,\alpha} \leq \left\| f \right\|_{p,\infty,\alpha}$. So that

$$\left(L^p_{uloc}\right)^\alpha(\mathbb{R}^d) \subset \left(L^p, \ell^\alpha\right)^\alpha(\mathbb{R}^d) = L^\alpha(\mathbb{R}^d). \tag{2.40}$$

The other cases are dealt with in the following proposition.

Proposition 2.16. *Let* $1 \leq \alpha \leq \infty$ *and* $1 \leq p \leq \infty$. *If* $\alpha < p$, *then*

$$\left(L^p_{uloc}\right)^\alpha(\mathbb{R}^d) = \{0\} .$$

Proof. Let $f \in \left(L^p_{uloc}\right)^\alpha(\mathbb{R}^d)$. Let us pose $A = \left\| f \right\|_{p,\infty,\alpha}$. For any real $r > 0$, we have:

$$\left\| f \right\|_{p,\infty} \leq \frac{A}{r^{d\left(\frac{1}{\alpha}-\frac{1}{p}\right)}}.$$

Next, $\lim_{r\to+\infty} \left\| f \right\|_{p,\infty} = 0$, since $\frac{1}{\alpha}-\frac{1}{p} > 0$. The case $p = +\infty$ is obsvious, because $\left\| f \right\|_{\infty,\infty} = \left\| f \right\|_\infty$. Thus $f = 0$. ■

Remark 2.3.2. Let $1 \leq p_1 \leq p \leq \alpha \leq \infty$.

- For any element $f \in L_0(\mathbb{R}^d)$,

$$\|f\|_{p_1,\infty,\alpha} \leq C \|f\|_{p,\infty,\alpha}, \tag{2.41}$$

where C is a constant independent of f. As a result,

$$\left(L^p_{uloc}\right)^\alpha (\mathbb{R}^d) \subset \left(L^{p_1}_{uloc}\right)^\alpha (\mathbb{R}^d). \tag{2.42}$$

- The space $\left(L^1_{uloc}\right)^\alpha (\mathbb{R}^d)$ is the classical space of Morrey.
- $L^\alpha(\mathbb{R}^d) \subset \left(L^p_{uloc}\right)^\alpha (\mathbb{R}^d) \subset \left(L^{p_1}_{uloc}\right)^\alpha (\mathbb{R}^d) \subset \left(L^1_{uloc}\right)^\alpha (\mathbb{R}^d)$.

2.4 $W^{1,p}_{uloc}(\mathbb{R}^d)$ Spaces, $(1 \leq p < \infty)$

After defining the spaces $L^p_{uloc}(\mathbb{R}^d)$ and give some properties, we will in this section discuss the Sobolev space $W^{1,p}_{uloc}(\mathbb{R}^d)$.

Definition 2.4.1. Let $1 \leq p < \infty$. The space $u \in W^{1,p}_{uloc}(\mathbb{R}^d)$ is a Sobolev space and we note

$$W^{1,p}_{uloc}(\mathbb{R}^d) = \left\{ u \in L^p_{uloc}(\mathbb{R}^d),\ \frac{\partial u}{\partial y_i} \in L^p_{uloc}(\mathbb{R}^d),\ \ 1 \leq i \leq d \right\}.$$

The space $W^{1,p}_{uloc}(\mathbb{R}^d)$ is associated with the norm

$$\|u\|_{W^{1,p}_{uloc}(\mathbb{R}^d)} = \left(\|u\|^p_{L^p_{uloc}(\mathbb{R}^d)} + \|\nabla u\|^p_{L^p_{uloc}(\mathbb{R}^d)} \right)^{\frac{1}{p}}. \tag{2.43}$$

is a Banach space.

Remark 2.4.1. As usual, in the case of où $p = 2$, we have $H^1_{uloc}(\mathbb{R}^d) = W^{1,2}_{uloc}(\mathbb{R}^d)$ and we define on $H^1_{uloc}(\mathbb{R}^d)$ the scalar product

$$(u, v)_{H^1_{uloc}(\mathbb{R}^d)} = (u, v)_{L^2_{uloc}(\mathbb{R}^d)} + \sum_{i=1}^{d} \left(\frac{\partial u}{\partial y_i}, \frac{\partial v}{\partial y_i} \right)_{L^2_{uloc}(\mathbb{R}^d)}, \quad \text{pour tout } u,\ v \in H^1_{uloc}(\mathbb{R}^d). \tag{2.44}$$

The associated norm

$$\|u\|_{H^1_{uloc}(\mathbb{R}^d)} = \left(\|u\|^2_{L^2_{uloc}(\mathbb{R}^d)} + \|\nabla u\|^2_{L^2_{uloc}(\mathbb{R}^d)} \right)^{\frac{1}{2}}. \tag{2.45}$$

Proposition 2.17. *We have the following statements:*

(1) The space $W^{1,p}_{uloc}(\mathbb{R}^d)$ is reflexive, $1 < p < \infty$.

(ii) The space $W^{1,p}_{uloc}(\mathbb{R}^d)$ is separable, $1 \leq p < \infty$.

2.5 Conclusion

The interest of $L^p_{uloc}(\mathbb{R}^d)$ spaces is highlighted by the fact that the spaces $(L^p, \ell^q)(\mathbb{R}^d)$ introduced by N. Wiener are a generalization, all the same the space $(L^p_{uloc})^\alpha(\mathbb{R}^d)$ highlighted in the penultimate block of this chapter is a sub-vector space of our study space and whose classical Lebesgue $L^\alpha(\mathbb{R}^d)$ spaces, $L^{(\alpha,+\infty)}(\mathbb{R}^d)$ Lorentz's and $(L^1_{uloc})^\alpha(\mathbb{R}^d)$ Morrey spaces are either subspaces or special cases. In the third chapter of our work, we will exploit the results of this study to solve a linear partial differential equation of the second order in divergence form in $W^{1,2}_{uloc}(\mathbb{R}^d)$.

⋆ ⋆ CHAPTER THREE ⋆ ⋆

APPLICATION IN SOBOLEV SPACE $W^{1,2}_{uloc}(\mathbb{R}^d)$

3.1 Problem statement

The main objective here is to solve in the sense of distributions the linear partial differential equation of the second order in the following divergence form:

$$-\text{div}(A\nabla u) + u = f + \text{div}F \quad \text{ in } \mathbb{R}^d, \tag{3.1}$$

where

$$f \in L^2_{uloc}(\mathbb{R}^d), F \in L^2_{uloc}(\mathbb{R}^d)^d \text{ et } A \in L^\infty(\mathbb{R}^d)^{d\times d} \text{ tels que } \alpha\,|\lambda|^2 \leq A(x)\lambda.\lambda \leq \beta\,|\lambda|^2\,. \tag{3.2}$$

for any $(x, \lambda) \in \mathbb{R}^d \times \mathbb{R}^d$, α et β are two positive real numbers. We show that (3.1) has a unique solution in $W^{1,2}_{uloc}(\mathbb{R}^d)$.

3.2 Existence and uniqueness results

We will need the Caccioppoli inequality formulated in the following theorem:

Theorem 3.1. (Caccioppoli's inequality)
Let u be a solution of (3.1). Then there exists a constant $C > 0$ (depending only on α, β and d) such that

$$\sup_{x\in\mathbb{R}^d} ⨍_{B_r(x)} \left(|\nabla u|^2 + |u|^2\right) \leq C + C \sup_{x\in\mathbb{R}^d} ⨍_{B_r(x)} \left(|f|^2 + |F|^2\right). \tag{3.3}$$

Proof. Let $\eta \in C_0^\infty(B_{2r}(x))$ be a regularising and truncated function such that $\eta = 1$ in $B_r(x)$,

$0 \leq \eta \leq 1$ and $|\nabla \eta| \leq C\, r^{-1}$. Taking $u\eta^2$ as the test function in (3.1), we have

$$\begin{aligned}\int_{B_{2r}(x)} \eta^2 A\nabla u.\nabla u + \int_{B_{2r}(x)} \eta^2 u^2 &= -2\int_{B_{2r}(x)} \eta u A\nabla u.\nabla \eta - 2\int_{B_{2r}(x)} \eta u F.\nabla \eta \\ &\quad - \int_{B_{2r}(x)} \eta^2 H.\nabla u + \int_{B_{2r}(x)} f\eta^2 u \qquad (3.4)\\ &= I_1 + I_2 + I_3 + I_4.\end{aligned}$$

The left-hand side of the above equality can be approximated by:

$$\alpha \int_{B_{2r}(x)} \eta^2 |\nabla u|^2 + \int_{B_r} \eta^2 |u|^2,$$

For the right-hand side, we use Young's inequality and the properties of the operator A.

$$|I_1| \leq \frac{\alpha}{8}\int_{B_{2r}(x)} \eta^2 |\nabla u|^2 + C\int_{B_{2r}(x)} |u|^2 |\nabla \eta|^2,$$

$$|I_2| \leq C\int_{B_{2r}(x)} \eta^2 |F|^2 + C\int_{B_{2r}(x)} |u|^2 |\nabla \eta|^2,$$

$$|I_3| \leq \frac{\alpha}{8}\int_{B_{2r}(x)} \eta^2 |\nabla u|^2 + C\int_{B_{2r}(x)} \eta^2 |F|^2,$$

$$|I_4| \leq \frac{1}{2}\int_{B_{2r}(x)} \eta^2 |u|^2 + \frac{1}{2}\int_{B_{2r}} \eta^2 |f|^2.$$

Finally, (3.4) becomes:

$$\int_{B_r(x)} (|\nabla u|^2 + |u|^2) \leq C\int_{B_{2r}(x)} (|\nabla u|^2 + |u|^2) + \frac{C}{r^2}\int_{B_{2r}(x)} |u|^2 + C\int_{B_{2r}(x)} (|f|^2 + |F|^2).$$

From ([9], Lemma 0.5), we infer that there exists a constant $C = C(\alpha, \beta, d)$ such that

$$\int_{B_r(x)} (|\nabla u|^2 + |u|^2) \leq \frac{C}{r^2}\int_{B_{2r}(x)} |u|^2 + \int_{B_{2r}(x)} (|f|^2 + |F|^2). \tag{3.5}$$

From (3.5), we have

$$⨍_{B_r(x)} (|\nabla u|^2 + |u|^2) \leq \frac{C}{r^2} ⨍_{B_{2r}(x)} |u|^2 + ⨍_{B_{2r}(x)} (|f|^2 + |F|^2). \tag{3.6}$$

Next, by substituting $\sup\limits_{x\in\mathbb{R}^d}$ in (3.6) and using the following inequality

$$\sup_{x\in\mathbb{R}^d} ⨍_{B_{2r}(x)} |v|^2 \leq C(d) \sup_{x\in\mathbb{R}^d} ⨍_{B_r(x)} |v|^2, \;\; \forall v \in L^2_{uloc}(\mathbb{R}^d)$$

It comes that

$$\sup_{x\in\mathbb{R}^d} ⨍_{B_r(x)} (|\nabla u|^2 + |u|^2) \leq \frac{C}{r^2} \sup_{x\in\mathbb{R}^d} ⨍_{B_r(x)} |u|^2 + C \sup_{x\in\mathbb{R}^d} ⨍_{B_r(x)} (|f|^2 + |F|^2). \tag{3.7}$$

In (3.7), we observe that if $r \geq \sqrt{2C}$, then the estimate (3.3) is satisfied. The case $1 \leq r \leq \sqrt{2C}$ is obtained from the case $r = 1$. ■

Theorem 3.2. Let $f \in L^2_{uloc}(\mathbb{R}^d)$ and $F \in L^2_{uloc}(\mathbb{R}^d)^d$. There exists an unique function $u \in W^{1,2}_{uloc}(\mathbb{R}^d)$ solution of (3.1). Furthermore, u holds :

$$\sup_{z\in\mathbb{R}^d} \fint_{B_r(z)} \left(|\nabla u|^2 + |u|^2\right) \leq C \sup_{z\in\mathbb{R}^d} \fint_{B_r(z)} \left(|f|^2 + |F|^2\right), \tag{3.8}$$

where $C = C(r, d, \alpha, \beta) > 0$ and $B_r(z) = B(z, r)$ denotes the open ball centred at z with radius r.

Proof. **1.** Existence.
Let $r > 0$ fixed and $v_r \in W^{1,2}_0(B_r)$ the unique solution of

$$-\text{div}(A\nabla v_r) + v_r = f + \text{div}F \quad \text{dans } B_r = B(0, r).$$

By adding the condition $v_r = 0$ on ∂B_r, we get that $(v_r)_r \in W^{1,2}_{loc}(\mathbb{R}^d)$. Let us show that the sequence $(v_r)_r$ is bounded in $W^{1,2}_{loc}(\mathbb{R}^d)$. We proceed as in ([11]). For the variational formulation of the above equation, we choose as test function, $\eta_z^2 v_r$, où $\eta_z = exp(-c\,|z|)$, for $z \in \mathbb{R}^d$ fixed, abd $c > 0$ chosen arbitrarily. we get

$$\begin{aligned}
-\int_{B_r} \text{div}(A\nabla v_r)\eta_z^2 v_r + \int_{B_r} \eta_z^2 v_r v_r &= \int_{B_r} f\eta_z^2 v_r + \int_{B_r} \text{div}F\eta_z^2 v_r \\
\int_{B_r} A\nabla v_r.\nabla(\eta_z^2 v_r) + \int_{B_r} \eta_z^2 v_r^2 &= \int_{B_r} h\eta_z^2 v_r + \int_{B_r} H.\nabla(\eta_z^2 v_r) \\
\int_{B_r} \eta_z^2 A\nabla v_r.\nabla v_r + \int_{B_r} \eta_z^2 v_r^2 &= -2\int_{B_r} \eta_z v_r A\nabla v_r.\nabla\eta_z - 2\int_{B_r} \eta_z v_r H.\nabla\eta_z \\
-\int_{B_r} \eta_z^2 H.\nabla v_r + \int_{B_r} h\eta_z^2 v_r & \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

The left-hand side of the above equality is bounded by:

$$\alpha \int_{B_r} \eta_z^2 \,|\nabla v_r|^2 + \int_{B_r} \eta_z^2 v_r^2,$$

For the right-hand side, we use Young's inequality and the properties on the operator A.

$$\begin{aligned}
|I_1| &= \left| -2\int_{B_r} \eta_z v_r A v_r.\nabla n_z \right| \\
&\leq \frac{1}{\epsilon}\int_{B_r} v_r^2\,|\nabla\eta_z|^2 + \epsilon\int_{B_r} \eta_z^2 A^2\,|\nabla v_r|^2 \qquad \text{for } \epsilon = \frac{k}{\alpha\beta} \\
|I_1| &\leq \frac{\alpha\beta}{k}\int_{B_r} v_r^2\,|\nabla\eta_z|^2 + \frac{\beta k}{\alpha}\int_{B_r} v_r\eta_z^2\,|\nabla v_r|^2.
\end{aligned}$$

$$|I_2| \leq \frac{\alpha\beta}{k}\int_{B_r} v_r^2\,|\nabla\eta_z|^2 + \frac{k}{\alpha\beta}\int_{B_r} \eta_z^2\,|F|^2$$

$$|I_3| \leq \frac{\beta k}{\alpha}\int_{B_r} \eta_z^2\,|\nabla v_r|^2 + \frac{\alpha}{4k\beta}\int_{B_r} \eta_z^2\,|F|^2$$

$$|I_4| \leq \frac{\alpha\beta c^2}{k}\int_{B_r} v_r^2\eta_z^2 + \frac{k}{4\alpha\beta c^2}\int_{B_r}\eta_z^2\,|f|^2\,,$$

where $k>0$ is chosen arbitrarily. Note that $|\nabla\eta_z| = c\eta_z$.

Thus, we have:

$$\int_{B_r}\eta_z^2\left(\alpha - 2\frac{\beta k}{\alpha}\right)|\nabla v_r|^2 + \int_{B_r}\eta_z^2\left(1-3\frac{\alpha\beta c^2}{k}\right)v_r^2 \leq \int_{B_r}\left[\left(\frac{k}{\alpha\beta}+\frac{\alpha}{4\beta k}\right)|F|^2 + \frac{k}{4\alpha\beta c^2}|f|^2\right]\eta_z^2$$

Next, for $k = \frac{\alpha^2}{4\beta}$ et $c = \frac{1}{2\beta}\left(\frac{\alpha}{6}\right)^{\frac{1}{2}}$, we have the following estimation:

$$\alpha\int_{B_r}\eta_z^2\,|\nabla v_r|^2 + \int_{B_r}\eta_z^2 v_r^2 \leq \int_{B_r}\left[\frac{3}{2}|f|^2 + \left(\frac{\alpha}{4\beta^2}+\frac{1}{\alpha}\right)|F|^2\right]\eta_z^2. \tag{3.9}$$

The inequality (3.9) shows that the sequence (v_r) is bounded in $W^{1,2}_{loc}(\mathbb{R}^d)$, indeed, for any compact $K\subset\mathbb{R}^d$, the left-hand side of the inequality (3.9) is bounded by: $C_K\left(\alpha\int_{B_r}|\nabla v_r|^2 + \int_{B_r}v_r^2\right)$ où $C_K = \min_K \eta_z^2 > 0$. While the right-hand side is approximated by $C\int_{\mathbb{R}^d}\eta_z^2$ where

$$C = \left(\frac{\alpha}{4\beta^2}+\frac{1}{\alpha}\right)\|F\|^2_{L^2_{uloc}} + \frac{3}{2}\|f\|^2_{L^2_{uloc}}\,.$$

Hence, there exists an unique sub-sequence (v_r) and a function $v\in W^{1,2}_{loc}(\mathbb{R}^d)$ such that the above-mentioned sub-sequence converges weakly to v in $W^{1,2}_{loc}(\mathbb{R}^d)$. This means that

$$v_r \to v \quad \text{dans } W^{1,2}_{loc}(\mathbb{R}^d) - \text{faible}.$$

Note that v is a weak solution of (3.1) in $\mathbb{R}^d$. By introducing the limit $\liminf\limits_{r\to\infty}$ in (3.9), we get:

$$\alpha\int_{\mathbb{R}^d}\eta_z^2\,|\nabla v_r|^2 + \int_{\mathbb{R}^d}\eta_z^2 v_r^2 \leq \int_{\mathbb{R}^d}\left[\frac{3}{2}|f|^2 + \left(\frac{\alpha}{4\beta}+\frac{1}{\alpha}|F|^2\right)\right]\eta_z^2 \tag{3.10}$$

Thus, we deduce from (3.10) that:

$$\sup_{z\in\mathbb{R}^d}⨍_{B_r(z)}\left(|\nabla v|^2+|v|^2\right)\leq C, \tag{3.11}$$

where $⨍_{B_r(z)} = \frac{1}{|B_r(z)|}\int_{B_r(z)}$ and C does not depend on z. For $r>1$, we have according to the Caccioppoli inequality,

$$⨍_{B_r(z)}|\nabla v|^2 + ⨍_{B_r(z)}|v|^2 \leq \frac{C}{r^2}⨍_{B_{2r}(z)}|\nabla v|^2 + C\left\{⨍_{B_r(z)}|f|^2 + ⨍_{B_r(z)}|F|^2\right\}. \tag{3.12}$$

for any $z\in\mathbb{R}^d$, where C depends only of d, α and β. Next, we have

$$\sup_{x\in\mathbb{R}^d}⨍_{B_{2r}}|v|^2 \leq C_d\sup_{x\in\mathbb{R}^d}⨍_{B_r}|v|^2\,. \tag{3.13}$$

Thus,

$$\sup_{x\in\mathbb{R}^d}⨍_{B_r}|\nabla v|^2 + \sup_{x\in\mathbb{R}^d}⨍_{B_r}|v|^2 \leq C\,r^{-2}\sup_{x\in\mathbb{R}^d}⨍_{B_r}|v|^2 + C\left\{\sup_{x\in\mathbb{R}^d}⨍_{B_r}\left(|f|^2+|F|^2\right)\right\}.$$

Ultimately, if $r\geq(2C)^{\frac{1}{2}}$, then (3.8). The case $r=1$ stems from [16].

2. Uniqueness.

   Proving the uniqueness of the solution amounts to considering (3.1) with $f = 0$ and $F = 0$. That is to say

$$-\text{div}(A\nabla v) + v = 0 \quad \text{dans } \mathbb{R}^d$$

According to the Caccioppoli inequality, we have:

$$\int_{B_r(z)} |\nabla v|^2 + \int_{B_r(z)} |v|^2 \leq \frac{C}{r^2} \int_{B_{2r}(z)} |v|^2 . \tag{3.14}$$

For $r \geq 1$. It stems from (3.14) :

$$\int_{B_r(z)} |v|^2 \leq \frac{C}{r^2} \int_{B_{2r}(z)} |v|^2 . \tag{3.15}$$

However, by virtue of (3.13) and (3.11), we have

$$\int_{B_{2r}(z)} |v|^2 \leq C.$$

Hence, (3.15) becomes

$$\int_{B_r(z)} |v|^2 \leq C\, r^{-2}, \quad \text{pour } r \geq 1. \tag{3.16}$$

Thus, by making $r \to +\infty$, we get $v = 0$ on $\mathbb{R}^d$. ■

Remark 3.2.1. The weak solution v of (3.1) given by the theorem 3.2 satisfies

$$\sup_{x\in\mathbb{R}^d} \left(\fint_{B(x,1)} |\nabla v|^p \right)^{\frac{1}{p}} \leq C \sup_{x\in\mathbb{R}^d} \left(\fint_{B(x,1)} |f|^2 \right)^{\frac{1}{2}} + C \sup_{x\in\mathbb{R}^d} \left(\fint_{B(x,1)} |F|^p \right)^{\frac{1}{p}} \tag{3.17}$$

$$\sup_{x\in\mathbb{R}^d} \left(\fint_{B(x,1)} |\nabla v|^q \right)^{\frac{1}{q}} \leq C \sup_{x\in\mathbb{R}^d} \left(\fint_{B(x,1)} |f|^2 \right)^{\frac{1}{2}} + C \sup_{x\in\mathbb{R}^d} \left(\fint_{B(x,1)} |F|^p \right)^{\frac{1}{p}} \tag{3.18}$$

for all $p > 2$, C depends only of d, α and β, where $\frac{1}{q} = \frac{1}{p} - \frac{1}{d}$ pour $d \geq 3$. If $d = 2$, the left-hand side of (3.18) can be replaced by $\|v\|_{L^\infty}$.

To have (3.17), we use the inverse estimation of Holder [9]: if v is a weak solution of $-\text{div}(A\nabla v) = f + \text{div}F$ dans $B_r = B(x,r)$ alors,

$$\left(\fint_{B_{\frac{r}{2}}} |\nabla v|^p \right)^{\frac{1}{p}} \leq \frac{C}{r} \left(\fint_{B_r} |v|^2 \right)^{\frac{1}{2}} + C \left(\fint_{B_r} |F|^p \right)^{\frac{1}{p}} + C\, r \left(\fint_{B_r} |f|^2 \right)^{\frac{1}{2}}$$

for all $p > 2$, C depends only of d, α and β.

In this part, we were asked to solve in the sense of distributions a linear second order partial differential equation in divergence form in the Sobolev type space $W^{1,2}_{uloc}(\mathbb{R}^d)$. The locally uniformly bounded solutions determined in this chapter play the role of the "correction term", which is very important in homogenization theory and is the solution of the associated correction problem defined by the gradient of the solution which is unique.

♣ Conclusion ♣

Having reached the end of this work whose objective was the study of locally uniformly bounded spaces and the application of the properties of these spaces to the search for locally uniformly bounded solutions of the equation (3.1) in the Sobolev type space $W^{1,2}_{uloc}(\mathbb{R}^d)$. For this reason, we have found it necessary to present some basic and useful results of functional analysis.

In this chapter, which is generally devoted to the preliminaries, we have focused mainly on the study of Lebesgue spaces and the space of sequences, where we have highlighted various properties relating to them. This allows us to introduce the second chapter of our work and to enter the heart of the matter. In this part, we defined the amalgam of $L^p(\mathbb{R}^d)$ and $ell^\infty(\mathbb{R}^d)$ (space of locally uniformly bounded functions) as that Banach space of (classes of) functions on a locally compact group, At the same time we have established some results in this space such as Holder's inequality, Young's inequality, the usual product and convolution to name a few and finally we have highlighted its relation with other spaces. Finally, we have shown that these spaces contain the Lebesgue $L^p(\mathbb{R}^d)$ spaces.

And finally in chapter three of this dissertation dedicated to the application as mentioned above, we first presented a preliminary result specific to our equation: the Caccioppoli inequality. Subsequently, we established through this inequality the existence and uniqueness theorem of the weak solution of our equation which is the main result of this thesis. In sum, this work opens the door to many applications especially in the theory of homogenization where the type of solutions determined in chapter three plays the role of a very important "corrective term" in this theory. It is in this perspective that we propose in our future work to solve problems in this type of space by means of homogenisation techniques, while proposing a corresponding numerical scheme.

♣ Bibliography ♣

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