CORRECTOR PROBLEM IN THE SPACE OF LOCALLY UNIFORMLY BOUNDED FUNCTIONS

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Abstract. Our aim in this paper is to provide a detailed proof to the existence of the corrector, based on the existence of the local weak solutions to linear problems. We find in the Sobolev type space $W^{1,2}_{uloc}(\mathbb{R}^d)$ the solution of a second order linear partial differential equation in divergence form. The obtained result constitutes an important step towards the numerical implementation of the results from the deterministic homogenization theory beyond the periodic setting.

1. Introduction and main result

The main objective of this current work is to solve in the sense of distributions a linear partial differential equation of the second order in divergence form. We provide an existence and uniqueness result of the weak solution by means of the Caccioppoli's Inequality specific to our problem in the Sobolev type space $W^{1,2}_{uloc}(\mathbb{R}^d)$ that is locally uniformly bounded energy function spaces.

The problem is stated as follows:

$$-\mathrm{div}(A\nabla u) + u = f + \mathrm{div}F \qquad \text{in } \mathbb{R}^d, \tag{1.1}$$

where $\mathbb{R}^d$ (integer $d \geq 1$) is the space of real numbers, the operator ∇ stands for the usual gradient, i.e. $\nabla = \left(\frac{\partial}{\partial x_i}\right)_{1\leq i\leq d}$, div denotes the divergence operator with respect to the variable x. The unknown is the function u and the coefficients in (1.1) are constrained as follows:

(**A1**) $A \in L^\infty(\mathbb{R}^d)^{d\times d}$ is a symmetric matrix verifying $\alpha\,|\lambda|^2 \leq A(x)\lambda\cdot\lambda \leq \beta\,|\lambda|^2$ for every $(x,\lambda) \in \mathbb{R}^d \times \mathbb{R}^d$, α and β are two positive real numbers.

(**A2**) $f \in L^2_{uloc}(\mathbb{R}^d)$ and $F \in L^2_{uloc}(\mathbb{R}^d)^d$.

The following theorem is the main result of the work.

Theorem 1.1. *Assume that* (**A1**)-(**A2**) *hold. There exists an unique function* $u \in W^{1,2}_{uloc}(\mathbb{R}^d)$ *solution of (1.1). Furthermore, the solution* u *satisfies the following uniform estimate:*

$$\sup_{z\in\mathbb{R}^d} \fint_{B_r(z)} \left(|\nabla u|^2 + |u|^2\right) \leq C \sup_{z\in\mathbb{R}^d} \fint_{B_r(z)} \left(|f|^2 + |F|^2\right), \tag{1.2}$$

where $C = C(r,d,\alpha,\beta) > 0$, $B_r(z) = B(z,r)$ *denotes the open ball centered at* z *with radius* r, *and* $\fint_{B_r(z)} = \frac{1}{|B_r(z)|}\int_{B_r(z)}$.

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Theorem 1.1 above establishes the existence of a distributional corrector and can be quite useful in the deterministic homogenization theory for a family of second order elliptic equations in divergence form with rapidly oscillating coefficients. It can enable us to easily find an approximate scheme for the homogenized coefficients, without smoothness assumption on the coefficients, which is a crucial step towards the numerical implementation of our results. Under additional condition, thanks once again to Theorem 1.1, we can also study the convergence rates in the asymptotic almost periodic setting. It is worth noticing that solving problem (1.1) in the sense of distributions lays the foundation to the study of regularity results in the general deterministic setting beyond the periodic framework. Thanks to the Caccioppoli inequality, we explicitly proved the important estimate (1.2), which is sharp compared to its counterpart in [22].

The rest of the paper is organized as follows. In Section 2, we present some preliminaries and we state some functional spaces. Section 3 is devoted to the proof of the main result by using some important estimates, thanks to Caciopolli Inequality.

2. Some functional spaces

Let us recall that by $L^p_{uloc}(\mathbb{R}^d)$ $(1 \leq p < \infty)$ is meant the subspace of $L^p_{loc}(\mathbb{R}^d)$ of those functions u such that

$$\sup_{x\in\mathbb{R}^d} \int_{B(x,1)} |u|^p \, dy < \infty$$

where $B(x,1)$ is the unit ball in $\mathbb{R}^d$ centered at x. Equipped with the norm

$$\|u\|_{L^p_{uloc}(\mathbb{R}^d)} = \sup_{x\in\mathbb{R}^d} \left(\int_{B(x,1)} |u|^p \, dy \right)^{\frac{1}{p}}, \tag{2.1}$$

$L^p_{uloc}(\mathbb{R}^d)$ is a Banach space. The Sobolev type space $L^p_{uloc}(\mathbb{R}^d)$ is actually the Wiener amalgam space $(L^p, \ell^\infty)(\mathbb{R}^d)$ introduced by Wiener [24]; see also [3, 11].

The norm (2.1) can be replaced by any of the following equivalent ones:

$$\|u\|_{L^p_{uloc}(\mathbb{R}^d)} \approx \sup_{\ell\in\mathbb{Z}^d} \left(\int_{\ell+(0,1)^d} |u|^p \, dy \right)^{\frac{1}{p}} \approx \sup_{\ell\in\mathbb{Z}^d} \left(\int_{\mathbb{R}^d} \varphi(y-\ell)^p \, |u(y)|^p \, dy \right)^{\frac{1}{p}}, \tag{2.2}$$

where φ is any nonnegative function in $\mathcal{C}^\infty_0(\mathbb{R}^d)$ such that $\sum_{k\in\mathbb{Z}^d} \varphi(y-k) \geq c_0 > 0$ for all $y \in \mathbb{R}^d$. We also set $L^\infty_{uloc}(\mathbb{R}^d) = L^\infty(\mathbb{R}^d)$.

With the identification $L^\infty_{uloc}(\mathbb{R}^d) = (L^p, \ell^\infty)(\mathbb{R}^d)$, we see that the properties of $L^\infty_{uloc}(\mathbb{R}^d)$ are now well known: see e.g. [3, Sections 6-7]. Let us recall one of its important properties that will be used in the sequel.

Lemma 2.1. *Let $f \in L^1(\mathbb{R}^d)$ and $g \in L^p_{uloc}(\mathbb{R}^d)$ $(1 \leq p < \infty)$. Then $f * g \in L^p_{uloc}(\mathbb{R}^d)$ and*

$$\|f * g\|_{L^p_{uloc}(\mathbb{R}^d)} \leq \|f\|_{L^1(\mathbb{R}^d)} \|g\|_{L^p_{uloc}(\mathbb{R}^d)}, \tag{2.3}$$

where $C = C(d,p) > 0$.

Proof. Since $f \in L^1(\mathbb{R}^d)$, it can be approximated by functions with compact support.

So $f*g$ will be well defined by

$$(f*g)(t)=\int_{\mathbb{R}^d} f(t-y)g(y)dy$$

one will show that (2.3) is satisfied. However (2.3) has been shown in [3, Section 7] and in [15, Chap 14, Sect. 14.1]. We repeat the proof for reader's convenience. To this end, we define $f_k (k\in\mathbb{Z}^d)$ by $f_k=f1_{k+(0,1)^d}$. Then $\sum_{k\in\mathbb{Z}^d}\|f_k\|_{L^1(\mathbb{R}^d)}=\|f\|_{L^1(\mathbb{R}^d)}$ and

$$\int_{x+(0,1)^d}|f_k*g|^p\,dy\leq\|f_k\|_{L^1(\mathbb{R}^d)}\int_{x-k+(0,4)^d}|g|^p\,dy.$$

Let $(x_i)_{1\leq i\leq 4^d}\subset\mathbb{R}^d$ be such that $B(0,4)\subset\cup_{i=1}^{4^d}\left(x_i+(0,1)^d\right)$. Then

$$\int_{x-k+(0,4)^d}|g|^p\,dy\leq\sum_{i=1}^{4^d}\int_{x-z+x_i+(0,1)^d}|g|^p\,dy\leq 4^d\sup_{x\in\mathbb{R}^d}\int_{x+(0,1)^d}|g|^p\,dy,$$

so that

$$\|f*g\|_{L^p_{uloc}(\mathbb{R}^d)}\leq\sum_{k\in\mathbb{Z}^d}\|f_k*g\|_{L^p_{uloc}(\mathbb{R}^d)}\leq 4^{\frac{d}{p}}\left(\sum_{k\in\mathbb{Z}^d}\|f_k\|_{L^1(\mathbb{R}^d)}\right)\|g\|_{L^p_{uloc}(\mathbb{R}^d)}$$

$$\leq 4^{\frac{d}{p}}\|f\|_{L^1(\mathbb{R}^d)}\|g\|_{L^p_{uloc}(\mathbb{R}^d)}$$

This completes the proof. □

We may also define the Sobolev type space $W^{1,p}_{uloc}(\mathbb{R}^d)$ accordingly.

$$W^{1,p}_{uloc}(\mathbb{R}^d)=\left\{u\in L^p_{uloc}(\mathbb{R}^d):\nabla u\in L^p_{uloc}(\mathbb{R}^d)^d\right\}$$

a Banach space with the norm

$$\|u\|_{W^{1,p}_{uloc}(\mathbb{R}^d)}=\left[\|u\|^p_{L^p_{uloc}(\mathbb{R}^d)}+\|\nabla u\|^p_{L^p_{uloc}(\mathbb{R}^d)}\right]^{\frac{1}{p}}.$$

3. Proof of the main result (Theorem 1.1)

We first need the Caccioppoli Inequality formulated as follows:

Theorem 3.1. *(Caccioppoli's Inequality)*
Let u be the solution of (1.1). Then there exists a constant $C>0$ (depending only on α, β and d) such that

$$\sup_{x\in\mathbb{R}^d}⨍_{B_r(x)}\left(|\nabla u|^2+|u|^2\right)\;\leq\;C\;+\;C\;\sup_{x\in\mathbb{R}^d}⨍_{B_r(x)}\left(|f|^2+|F|^2\right). \tag{3.1}$$

Proof. Let $\eta \in C_0^\infty(B_{2r}(x))$ be a regularising and truncated function such that $\eta = 1$ in $B_r(x)$, $0 \leq \eta \leq 1$ and $|\nabla \eta| \leq C\ r^{-1}$. Taking $u\eta^2$ as the test function in (1.1), we have

$$\begin{aligned}\int_{B_{2r}(x)} \eta^2 A\nabla u \cdot \nabla u + \int_{B_{2r}(x)} \eta^2 u^2 &= -2\int_{B_{2r}(x)} \eta u A\nabla u \cdot \nabla\eta - 2\int_{B_{2r}(x)} \eta u F \cdot \nabla \eta \\ &\quad - \int_{B_{2r}(x)} \eta^2 H \cdot \nabla u + \int_{B_{2r}(x)} f\eta^2 u \\ &= I_1 + I_2 + I_3 + I_4.\end{aligned} \tag{3.2}$$

The left-hand side of the above equality can be approximated by:

$$\alpha \int_{B_{2r}(x)} \eta^2 |\nabla u|^2 + \int_{B_r} \eta^2 |u|^2 .$$

For the right-hand side, we use Young's Inequality and the properties of the operator A.

$$|I_1| \ \leq \ \frac{\alpha}{8}\int_{B_{2r}(x)} \eta^2 |\nabla u|^2 + C\ \int_{B_{2r}(x)} |u|^2 |\nabla \eta|^2 ,$$

$$|I_2| \leq C\ \int_{B_{2r}(x)} \eta^2 |F|^2 + C\ \int_{B_{2r}(x)} |u|^2 |\nabla \eta|^2 ,$$

$$|I_3| \leq \frac{\alpha}{8}\int_{B_{2r}(x)} \eta^2 |\nabla u|^2 + C\ \int_{B_{2r}(x)} \eta^2 |F|^2 ,$$

$$|I_4| \leq \frac{1}{2}\int_{B_{2r}(x)} \eta^2 |u|^2 + \frac{1}{2}\int_{B_{2r}} \eta^2 |f|^2 .$$

Finally, (3.2) becomes:

$$\int_{B_r(x)} (|\nabla u|^2 + |u|^2) \leq C\ \int_{B_{2r}(x)} (|\nabla u|^2 + |u|^2) + \frac{C}{r^2}\ \int_{B_{2r}(x)} |u|^2 + C\ \int_{B_{2r}(x)} (|f|^2 + |F|^2).$$

From [12, Lemma 0.5], we infer that there exists a constant $C = C(\alpha, \beta, d)$ such that

$$\int_{B_r(x)} (|\nabla u|^2 + |u|^2) \leq \frac{C}{r^2}\ \int_{B_{2r}(x)} |u|^2 + \int_{B_{2r}(x)} (|f|^2 + |F|^2). \tag{3.3}$$

which implies

$$⨍_{B_r(x)} (|\nabla u|^2 + |u|^2) \leq \frac{C}{r^2} ⨍_{B_{2r}(x)} |u|^2 + ⨍_{B_{2r}(x)} (|f|^2 + |F|^2). \tag{3.4}$$

Next, by substituting $\sup\limits_{x\in\mathbb{R}^d}$ in (3.4) and using the following inequality

$$\sup_{x\in\mathbb{R}^d} ⨍_{B_{2r}(x)} |v|^2 \leq C(d)\ \sup_{x\in\mathbb{R}^d} ⨍_{B_r(x)} |v|^2 , \quad \forall v \in L^2_{uloc}(\mathbb{R}^d),$$

it comes that

$$\sup_{x\in\mathbb{R}^d} ⨍_{B_r(x)} (|\nabla u|^2 + |u|^2) \leq \frac{C}{r^2}\ \sup_{x\in\mathbb{R}^d} ⨍_{B_r(x)} |u|^2 + C\ \sup_{x\in\mathbb{R}^d} ⨍_{B_r(x)} (|f|^2 + |F|^2). \tag{3.5}$$

In (3.5), we observe that if $r \geq \sqrt{2C}$, then the estimate (3.1) is satisfied. The case $1 \leq r \leq \sqrt{2C}$ is obtained from the case $r = 1$. $\square$

Proof. **Proof of Theorem 1.1**

(1) **Existence**.

Let $r > 0$ be fixed and $v_r \in W_0^{1,2}(B_r)$ the unique solution of

$$-\mathrm{div}(A\nabla v_r) + v_r = f + \mathrm{div}F \quad \text{in } B_r = B(0, r).$$

By adding the condition $v_r = 0$ on ∂B_r, we can prove that $(v_r)_r \in W_{loc}^{1,2}(\mathbb{R}^d)$. Let us show that the sequence $(v_r)_r$ is bounded in $W_{loc}^{1,2}(\mathbb{R}^d)$. We proceed exactly as in [11]. For the variational formulation of the above equation, we choose $\eta_z^2 v_r$ as test function, where $\eta_z = exp(-c\,|z|)$, for $z \in \mathbb{R}^d$ freely fixed and $c > 0$ arbitrarily chosen. We get

$$\begin{aligned}
-\int_{B_r} \mathrm{div}(A\nabla v_r)\eta_z^2 v_r + \int_{B_r} \eta_z^2 v_r v_r &= \int_{B_r} f\eta_z^2 v_r + \int_{B_r} \mathrm{div}F \eta_z^2 v_r \\
\int_{B_r} A\nabla v_r.\nabla(\eta_z^2 v_r) + \int_{B_r} \eta_z^2 v_r^2 &= \int_{B_r} h\eta_z^2 v_r + \int_{B_r} H.\nabla(\eta_z^2 v_r) \\
\int_{B_r} \eta_z^2 A\nabla v_r.\nabla v_r + \int_{B_r} \eta_z^2 v_r^2 &= -2\int_{B_r} \eta_z v_r A\nabla v_r \cdot \nabla\eta_z - 2\int_{B_r} \eta_z v_r H.\nabla\eta_z \\
-\int_{B_r} \eta_z^2 H.\nabla v_r + \int_{B_r} h\eta_z^2 v_r & \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

The left-hand side of the above equality is bounded by:

$$\alpha \int_{B_r} \eta_z^2 \left|\nabla v_r\right|^2 + \int_{B_r} \eta_z^2 v_r^2.$$

For the right-hand side, the Young inequality and the properties of the operator A give rise to the ensuing estimates.

$$\begin{aligned}
|I_1| &= \left| -2\int_{B_r} \eta_z v_r A v_r \cdot \nabla n_z \right| \\
&\leq \frac{1}{\epsilon}\int_{B_r} v_r^2 \left|\nabla\eta_z\right|^2 + \epsilon\int_{B_r} \eta_z^2 A^2 \left|\nabla v_r\right|^2 \qquad \text{for } \epsilon = \frac{k}{\alpha\beta} \\
|I_1| &\leq \frac{\alpha\beta}{k}\int_{B_r} v_r^2 \left|\nabla\eta_z\right|^2 + \frac{\beta k}{\alpha}\int_{B_r} v_r\eta_z^2 \left|\nabla v_r\right|^2.
\end{aligned}$$

$$|I_2| \leq \frac{\alpha\beta}{k}\int_{B_r} v_r^2 \left|\nabla\eta_z\right|^2 + \frac{k}{\alpha\beta}\int_{B_r} \eta_z^2 \left|F\right|^2$$

$$|I_3| \leq \frac{\beta k}{\alpha}\int_{B_r} \eta_z^2 \left|\nabla v_r\right|^2 + \frac{\alpha}{4k\beta}\int_{B_r} \eta_z^2 \left|F\right|^2$$

$$|I_4| \leq \frac{\alpha\beta c^2}{k}\int_{B_r} v_r^2\eta_z^2 + \frac{k}{4\alpha\beta c^2}\int_{B_r} \eta_z^2 \left|f\right|^2,$$

where $k > 0$ is arbitrarily chosen . Note that $|\nabla \eta_z| = c\eta_z$.

Thus, we have:

$$\int_{B_r} \eta_z^2 \left(\alpha - 2\frac{\beta k}{\alpha}\right) |\nabla v_r|^2 + \int_{B_r} \eta_z^2 \left(1 - 3\frac{\alpha\beta c^2}{k}\right) v_r^2 \leq \int_{B_r} \left[\left(\frac{k}{\alpha\beta} + \frac{\alpha}{4\beta k}\right) |F|^2 + \frac{k}{4\alpha\beta c^2} |f|^2\right] \eta_z^2.$$

Next, for $k = \frac{\alpha^2}{4\beta}$ and $c = \frac{1}{2\beta}\left(\frac{\alpha}{6}\right)^{\frac{1}{2}}$, we have the following estimation:

$$\alpha \int_{B_r} \eta_z^2 |\nabla v_r|^2 + \int_{B_r} \eta_z^2 v_r^2 \leq \int_{B_r} \left[\frac{3}{2} |f|^2 + \left(\frac{\alpha}{4\beta^2} + \frac{1}{\alpha}\right) |F|^2\right] \eta_z^2. \tag{3.6}$$

Inequality (3.6) shows that the sequence (v_r) is bounded in $W^{1,2}_{loc}(\mathbb{R}^d)$. Indeed, for any compact $K \subset \mathbb{R}^d$, the left-hand side of Inequality (3.6) is bounded by

$$C_K \left(\alpha \int_{B_r} |\nabla v_r|^2 + \int_{B_r} v_r^2\right),$$

where $C_K = \min_K \eta_z^2 > 0$, while the right-hand side is approximated by $C \int_{\mathbb{R}^d} \eta_z^2$ where

$$C = \left(\frac{\alpha}{4\beta^2} + \frac{1}{\alpha}\right) \|F\|^2_{L^2_{uloc}} + \frac{3}{2} \|f\|^2_{L^2_{uloc}}.$$

Hence, there exists an unique sub-sequence (v_r) and a function $v \in W^{1,2}_{loc}(\mathbb{R}^d)$ such that the above-mentioned sub-sequence weakly converges to v in $W^{1,2}_{loc}(\mathbb{R}^d)$. This means that

$$v_r \to v \quad \text{in } W^{1,2}_{loc}(\mathbb{R}^d) - \text{weak}.$$

Note that v is a weak solution of (1.1) in $\mathbb{R}^d$. By introducing the limit $\varliminf_{r\to\infty}$ in (3.6), we get:

$$\alpha \int_{\mathbb{R}^d} \eta_z^2 |\nabla v_r|^2 + \int_{\mathbb{R}^d} \eta_z^2 v_r^2 \leq \int_{\mathbb{R}^d} \left[\frac{3}{2} |f|^2 + \left(\frac{\alpha}{4\beta} + \frac{1}{\alpha} |F|^2\right)\right] \eta_z^2. \tag{3.7}$$

Thus, we deduce from (3.7) that:

$$\sup_{z\in\mathbb{R}^d} \fint_{B_r(z)} \left(|\nabla v|^2 + |v|^2\right) \leq C, \tag{3.8}$$

where $\fint_{B_r(z)} = \frac{1}{|B_r(z)|} \int_{B_r(z)}$ and C does not depend on z. For $r > 1$, according to the Caccioppoli inequality, we have:

$$\fint_{B_r(z)} |\nabla v|^2 + \fint_{B_r(z)} |v|^2 \leq \frac{C}{r^2} \fint_{B_{2r}(z)} |\nabla v|^2 + C \left\{\fint_{B_r(z)} |f|^2 + \fint_{B_r(z)} |F|^2\right\}, \tag{3.9}$$

for any $z \in \mathbb{R}^d$, where C depends only of d, α and β. Next, we have

$$\sup_{x\in\mathbb{R}^d} \fint_{B_{2r}} |v|^2 \leq C_d \sup_{x\in\mathbb{R}^d} \fint_{B_r} |v|^2. \tag{3.10}$$

Thus,

$$\sup_{x\in\mathbb{R}^d} \fint_{B_r} |\nabla v|^2 + \sup_{x\in\mathbb{R}^d} \fint_{B_r} |v|^2 \leq C\, r^{-2} \sup_{x\in\mathbb{R}^d} \fint_{B_r} |v|^2 + C \left\{\sup_{x\in\mathbb{R}^d} \fint_{B_r} \left(|f|^2 + |F|^2\right)\right\}.$$

Ultimately, if $r \geq (2C)^{\frac{1}{2}}$, then (1.2) holds. The case $r = 1$ stems from [14].

(2) **Uniqueness**.
Proving the uniqueness of the solution amounts to considering (1.1) with $f = 0$ and $F = 0$. That is to say

$$-\mathrm{div}(A\nabla v) + v = 0 \quad \text{ in } \mathbb{R}^d$$

According to the Caccioppoli Inequality, we have:

$$\int_{B_r(z)} |\nabla v|^2 + \int_{B_r(z)} |v|^2 \leq \frac{C}{r^2} \int_{B_{2r}(z)} |v|^2 . \tag{3.11}$$

For $r \geq 1$. It stems from (3.11) :

$$\int_{B_r(z)} |v|^2 \leq \frac{C}{r^2} \int_{B_{2r}(z)} |v|^2 . \tag{3.12}$$

However, by virtue of (3.10) and (3.8), we have

$$\int_{B_{2r}(z)} |v|^2 \leq C.$$

Hence, (3.12) becomes

$$\int_{B_r(z)} |v|^2 \leq C\ r^{-2}, \quad \text{ for } r \geq 1. \tag{3.13}$$

Thus, by making $r \to +\infty$, we get $v = 0$ on $\mathbb{R}^d$.

$\square$

It is worth noticing that the weak solution v of (1.1) given by the theorem 1.1 satisfies

$$\sup_{x\in\mathbb{R}^d} \left(\fint_{B(x,1)} |\nabla v|^p \right)^{\frac{1}{p}} \leq C \sup_{x\in\mathbb{R}^d} \left(\fint_{B(x,1)} |f|^2 \right)^{\frac{1}{2}} + C \sup_{x\in\mathbb{R}^d} \left(\fint_{B(x,1)} |F|^p \right)^{\frac{1}{p}} \tag{3.14}$$

$$\sup_{x\in\mathbb{R}^d} \left(\fint_{B(x,1)} |\nabla v|^q \right)^{\frac{1}{q}} \leq C \sup_{x\in\mathbb{R}^d} \left(\fint_{B(x,1)} |f|^2 \right)^{\frac{1}{2}} + C \sup_{x\in\mathbb{R}^d} \left(\fint_{B(x,1)} |F|^p \right)^{\frac{1}{p}} \tag{3.15}$$

for all $p > 2$, C depends only of d, α and β, where $\frac{1}{q} = \frac{1}{p} - \frac{1}{d}$ pour $d \geq 3$. If $d = 2$, the left-hand side of (3.15) can be replaced by $\|v\|_{L^\infty}$.

To have (3.14), we use the inverse estimation of Holder [12]: if v is a weak solution of

$$-\mathrm{div}(A\nabla v) = f + \mathrm{div}F \ \text{ in } \ B_r = B(x,r) \text{ then,}$$

$$\left(\fint_{B_{\frac{r}{2}}} |\nabla v|^p \right)^{\frac{1}{p}} \leq \frac{C}{r} \left(\fint_{B_r} |v|^2 \right)^{\frac{1}{2}} + C \left(\fint_{B_r} |F|^p \right)^{\frac{1}{p}} + C\ r \left(\fint_{B_r} |f|^2 \right)^{\frac{1}{2}}$$

for all $p > 2$, C depends only of d, α and β.

Conflict of interests.
The authors declare that there is no conflict of interest regarding the publication of this paper.

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