Modeling the dynamics of a forest environment: role of water cycle

Biocore seminar

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Introduction

Forests are part of the more or less complex ecosystems of the planet because they are full of organisms of small and large scales, namely trees, animals, and bacteria.

To understand the functioning of forest ecosystems, it is useful to focus on the influence of certain factors in the forest environment. We refer to the impact of climate change, water resources, and deforestation.

Introduction

Motivations

Scientists, specifically biologists, do not easily distinguish tree categories in the field based on their density.

Objective

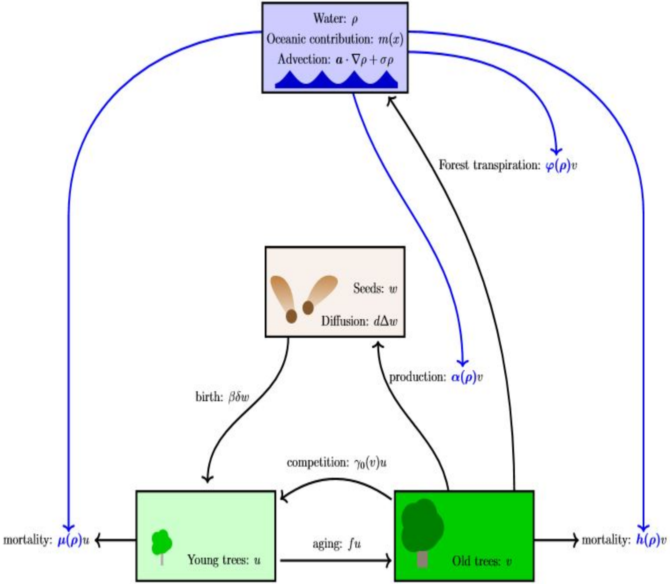
In this work, we aim to adapt the mathematical model proposed by Cantin et al [1] based on a reaction-diffusion-advection system by taking into account the effects of atmospheric activity and water resources.

Background

- A survey about the management of the forests [3],
- The impact of climate change and some extreme events on the forest.

Antonovsky et al [2] introduced a simple model of age structure dynamics of the monospecies system, and in 2021 Cantin et al [1] proposed a novel age model to study the dynamics of a forest ecosystems.

Problem statement



Problem statement

It is described by the following reaction-diffusion advection system of four partial differential equations.

$$\left\{\begin{array}{rcl} a.\nabla\rho(t,x) & = & -\sigma\rho+\varphi(\rho)v, \\ \dfrac{\partial u}{\partial t}(t,x) & = & \beta\delta\omega-\gamma(v,\rho)u-fu, \\ \dfrac{\partial v}{\partial t}(t,x) & = & fu-h(\rho)v, \\ \dfrac{\partial \omega}{\partial t}(t,x) & = & d\Delta\omega-\beta\omega+\alpha(\rho)v, \end{array}\right. \tag{1}$$

Problem statement

Antonovsky et al assumed that the **overall tree mortality rate** $\gamma(v, \rho)$ can be defined as

$$\gamma(v, \rho) = \gamma_0(v) + \mu(\rho). \tag{2}$$

Here, the function $\gamma_0(v)$ refers to the **competition** between young and old trees. It is defined by a quadratic form:

$$\gamma_0(v) = r(v - b)^2 + c. \tag{3}$$

The competition term γ_0 is highlighted by considering life resources (water, light).

Problem statement

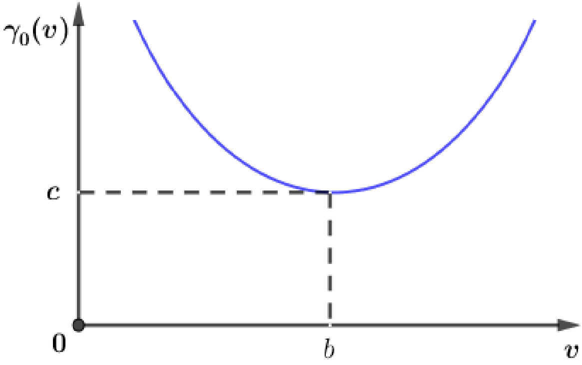


Figure 1 : Illustration of tree competition.

Abstract forest

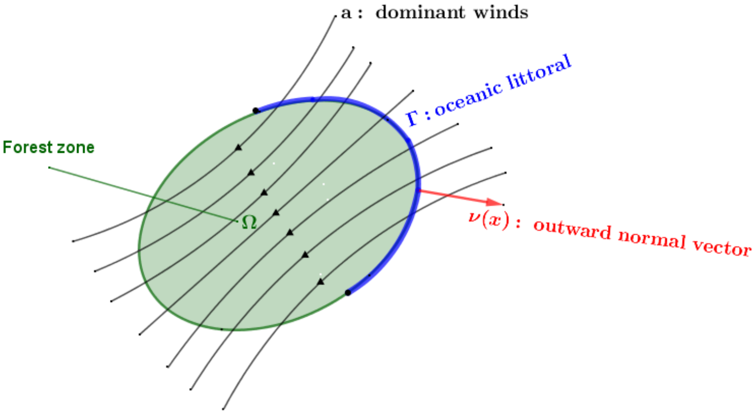


Figure 2 : Geographic representation of a forest area [1].

Next, we considered that the oceanic littoral Γ is defined by:

$$\Gamma = \{x \in \mathbb{R}, \quad a(x).\nu(x) < 0\}.$$

Unstructured age model

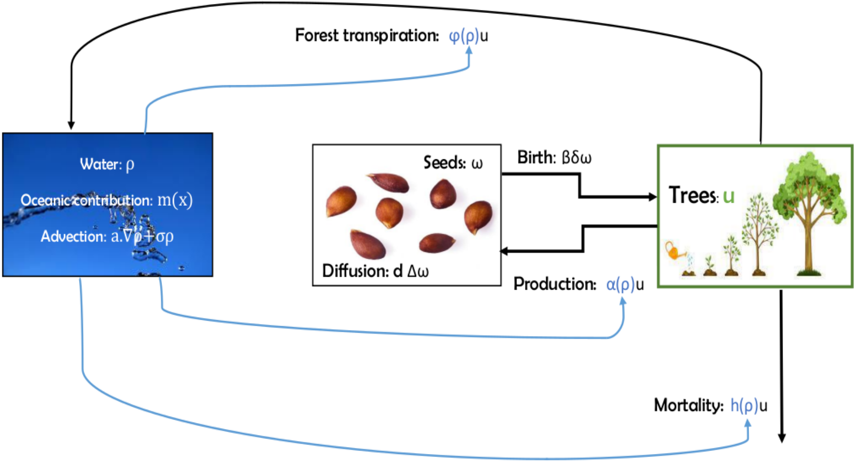


Figure 3 : A compartmental model of our forest ecosystem.

Model formulation

Next, we have:

$$\begin{cases} a.\nabla\rho(t,x) & = & -\sigma\rho + \varphi(\rho)u, \\ \dfrac{\partial u}{\partial t}(t,x) & = & \beta\delta\omega - \gamma(\rho)u, \\ \dfrac{\partial \omega}{\partial t}(t,x) & = & d\Delta\omega - \beta\omega + \alpha(\rho)u. \end{cases} \tag{4}$$

In this case, $\gamma(\rho)$ represents the **overall tree mortality** and it is described by:

$$\gamma(\rho) = k + h(\rho), \tag{5}$$

Model formulation

Conditions on system (4)

Neumann boundary and initial conditions associated to system (4) are given as follows:

$$
\begin{cases}
\rho(t,x) = m(x), & t>0, \quad x \in \Gamma, \\[2ex]
\dfrac{\partial \omega}{\partial \nu}(t,x) = 0, & t>0, \quad x \in \partial\Omega, \\[2ex]
u(0,x) = u_0(x), \qquad \omega(0,x) = \omega_0(x), & x \in \Omega.
\end{cases}
\tag{6}
$$

Model reduction

Here, we aim to parametrize the **advection equation** to reduce the system (4) into a reaction-diffusion system.

For that, we introduce and show the well-posedness of the following operator:

$$\begin{aligned} \psi : L^{\infty}_{+}(\Omega) \quad &\longrightarrow \quad L^{\infty}_{+}(\Omega), \\ u \quad &\longmapsto \quad \rho. \end{aligned} \tag{7}$$

where $\rho = \psi(u)$ is the **solution** of advection equation.

Model reduction

Theorem

For $x \in \Omega$ almost everywhere (a.e) and $u \in L_+^\infty(\Omega)$, the defined operator ψ in (7) exists and it is uniquely determined along the characteristic lines of the advection field a by:

$$\psi(u)(x) = m(\zeta_1(x))e^{-\sigma\zeta_2(x)} + \int_0^{\zeta_2} \varphi\left(\tilde{\rho}(\zeta_1(x), \tau)\right)\ \tilde{u}(\zeta_1(x), \tau)e^{-\sigma(\zeta_2(x)-\tau)}, \tag{8}$$

where $(x_0, s) = (\zeta_1(x), \zeta_2(x))$. Furthermore, the operator ψ is continuous in $L_+^\infty(\Omega)$ and we have:

$$\|\psi(u+h) - \psi(u)\| \leq \|h\|_\infty \times \frac{\varphi_0}{\sigma} e^{\varphi_0 \bar{S}\|u\|_\infty}, \ \forall\ u, h \in L_+^\infty(\Omega). \tag{9}$$

Model reduction

According to theorem above, system (4) becomes:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) &= & \beta\delta\omega - \gamma(\psi(u))u, & t>0, \quad x\in\Omega, \\ \dfrac{\partial \omega}{\partial t}(t,x) &= & d\Delta\omega - \beta\omega + \alpha(\psi(u))u, & t>0, \quad x\in\Omega, \\ \dfrac{\partial \omega}{\partial \nu}(t,x) &= & 0, & t>0, \quad x\in\partial\Omega, \end{cases} \tag{10}$$

where $\psi(u)$ models the dependence of water resource in the tree life process.

Model reduction

System (10) can be written as follows:

$$\begin{pmatrix} \dfrac{\partial u}{\partial t} \\ \\ \dfrac{\partial \omega}{\partial t} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & \Lambda \end{pmatrix} \begin{pmatrix} u \\ \omega \end{pmatrix} = \begin{pmatrix} \beta\delta\omega - \gamma(\psi(u))u + u \\ \\ \alpha(\psi(u))u \end{pmatrix}, \qquad (11)$$

Model reduction

We now have an **abstract Cauchy** Problem

$$\begin{cases} \dfrac{dU}{dt} + AU = F(U), & t > 0, \\ U(0) = U_0, & U_0 \in \mathcal{Z}, \end{cases} \tag{12}$$

where $\Lambda = -d\Delta + \beta$ a linear operator in $L^2(\Omega)$, $A = \mathrm{diag}(1, \Lambda)$, and

$$U = (u, \omega)^t \in \mathcal{D}(A^\eta), \qquad F(U) = \begin{bmatrix} \beta\delta\omega - \gamma(\psi(u))u + u \\ \alpha(\psi(u))u \end{bmatrix}.$$

Model analysis

Theorem

For any initial condition $U_0 \in \mathcal{Z}$, the Cauchy problem (12) possesses a **unique local** solution in time $U = (u, \omega)^t$ defined on $Y = [0, T_{U_0}]$ with

$$\begin{cases} u \in \mathcal{C}\left(Y, L^\infty(\Omega)\right) \cap \mathcal{C}^1\left((0, T_{U_0}], L^\infty(\Omega)\right), \\ \omega \in \mathcal{C}\left((0, T_{U_0}], \mathcal{D}(\Lambda)\right) \cap \mathcal{C}\left(Y, L^2(\Omega)\right) \cap \mathcal{C}^1\left((0, T_{U_0}], L^2(\Omega)\right), \end{cases} \tag{13}$$

where $T_{U_0} = \mathrm{Cte}(\|U_0\|_X) > 0$. Furthermore, the local solution U satisfies

$$t \|AU(t)\|_X + \|U(t)\|_X \leq T_{U_0}, \qquad 0 < t \leq T_{U_0}. \tag{14}$$

Model analysis

Theorem

Let $0 \leq u_0 \in L^\infty(\Omega)$ and $0 \leq \omega_0 \in L^2(\Omega)$. System (10) admits a **unique non-negative local** solution such that:

$$\begin{cases} 0 \leq u \in \mathcal{C}\left(Y, L^\infty(\Omega)\right) \cap \mathcal{C}^1\left((0, T_{U_0}], L^\infty(\Omega)\right), \\ 0 \leq \omega \in \mathcal{C}\left((0, T_{U_0}], \mathcal{D}(\Lambda)\right) \cap \mathcal{C}\left(Y, L^2(\Omega)\right) \cap \mathcal{C}^1\left((0, T_{U_0}], L^2(\Omega)\right), \end{cases} \tag{15}$$

Non-negativity of the solution

Hint for the proof

We introduced the following cut-off function ϑ defined by:

$$\vartheta(\hat{u}) = \begin{cases} \hat{u} & \text{if} \quad \hat{u} \geq 0, \\ 0 & \text{if} \quad \hat{u} < 0. \end{cases} \qquad \text{and} \qquad \vartheta(\hat{\omega}) = \begin{cases} \hat{\omega} & \text{if} \quad \hat{\omega} \geq 0, \\ 0 & \text{if} \quad \hat{\omega} < 0. \end{cases} \tag{16}$$

Global solutions

Let $0 \leq u_0 \in L^\infty(\Omega)$, $0 \leq \omega_0 \in L^2(\Omega)$, $\eta > 0$, and $0 < t \leq T_{U_0}$. Under **Caratheodory properties**, since the local solution $U = (\rho, u, \omega)^t$ of (4) in the function space

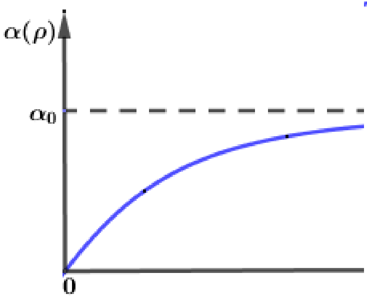
$$\begin{cases} 0 \leq \rho,\ \ u \in \mathcal{C}\left(Y, L^\infty(\Omega)\right) \cap \mathcal{C}^1\left((0, T_{U_0}], L^\infty(\Omega)\right), \\ 0 \leq \omega \in \mathcal{C}\left((0, T_{U_0}], \mathcal{D}(\Lambda)\right) \cap \mathcal{C}\left(Y, L^2(\Omega)\right) \cap \mathcal{C}^1\left((0, T_{U_0}], L^2(\Omega)\right). \end{cases}$$

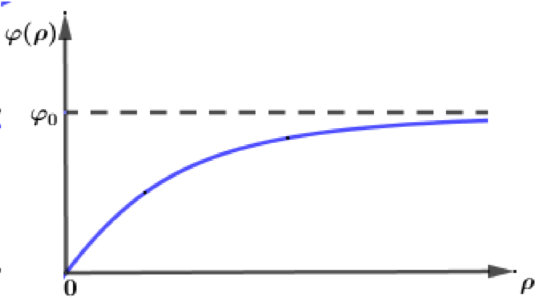
satisfies (14). More precisely, the following estimate is satisfied

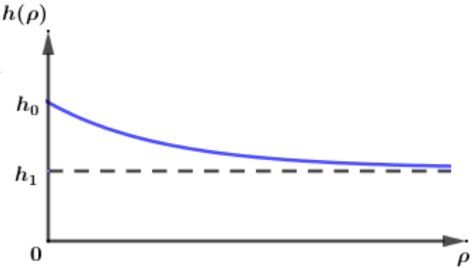
$$\|u(t)\|_{L^\infty(\Omega)} + \|\omega(t)\|_{L^2(\Omega)} \leq C\left[e^{-t\eta}\left(\|u_0\|_{L^\infty(\Omega)} + \|\omega_0\|_{L^2(\Omega)}\right)\right]. \quad (17)$$

Then, the reaction-diffusion advection system (4) admits a **unique global** solution $U = (\rho, u, \omega)^t$.

Stability analysis







Estimated functions

$$\alpha(\rho) = \frac{\alpha_0 \rho}{1+\rho}, \qquad \varphi(\rho) = \frac{\varphi_0 \rho}{1+\rho}, \qquad h(\rho) = \frac{h_0 + h_1 \rho}{1+\rho}. \tag{18}$$

Let $\bar{U} = (\bar{\rho}, \bar{u}, \bar{\omega})^t$ a stationary solution of system (4). We have

$$\begin{cases} \bar{\rho} = m(x), \\ a.\nabla \bar{\rho} = -\sigma \bar{\rho} + \varphi(\bar{\rho})\bar{u}, \\ \dfrac{\partial \bar{u}}{\partial t} = \beta\delta\bar{\omega} - \gamma(\bar{\rho})\bar{u}, \\ \dfrac{\partial \bar{\omega}}{\partial t} = d\Delta\bar{\omega} - \beta\bar{\omega} + \alpha(\bar{\rho})\bar{u}. \end{cases} \tag{19}$$

Stability analysis

1. **Oceanic contribution is vanishing** (m(x)=0). We have:

$$\begin{cases} \bar{\rho} = 0, \\ \varphi(0)\bar{u} = 0, \\ \beta\delta\bar{\omega} = \gamma(0)\bar{u}, \\ \beta\bar{\omega} = \alpha(0)\bar{u}. \end{cases} \tag{20}$$

Hence, $\bar{u} = 0$, and also $\bar{\omega} = 0$. Thus, the **trivial solution** $\bar{U} = (\bar{\rho}, \bar{u}, \bar{\omega}) = (0, 0, 0)$ is a unique stationary homogeneous solution of system (4).

Stability analysis

Proposition

Let us assume that the regular function is vanishing (i.e. $\rho = m(x) = 0$), for all $x \in \Gamma$, then system (4) possesses a unique stationary homogeneous solution $\bar{U} = (\bar{\rho}, \bar{u}, \bar{\omega}) = (0, 0, 0)$.

2. **Oceanic contribution is not vanishing** $(m(x) > 0)$. We have:

$$\begin{cases} \bar{\rho} = \bar{m}, \\ \sigma \bar{m} = \varphi(\bar{m})\bar{u}, \\ \beta\delta\bar{\omega} = \gamma(\bar{m})\bar{u}, \\ \beta\bar{\omega} = \alpha(\bar{m})\bar{u}. \end{cases} \tag{21}$$

After solving the latter system, we get:

$$\bar{u} = \frac{\sigma \bar{m}}{\varphi(\bar{m})}, \qquad \text{and} \qquad \bar{\omega} = \frac{\alpha(\bar{m})}{\beta}\bar{u}. \tag{22}$$

Stability analysis

Since

$$\alpha(\rho) = \frac{\alpha_0 \rho}{1+\rho}, \qquad \varphi(\rho) = \frac{\varphi_0 \rho}{1+\rho}, \qquad h(\rho) = \frac{h_0 + h_1 \rho}{1+\rho}.$$

We finally get:

$$\bar{\rho} = \bar{m}, \qquad \bar{u} = \frac{\sigma}{\varphi_0}(1+\bar{m}), \qquad \bar{\omega} = \frac{\alpha_0}{\beta}\frac{\sigma}{\varphi_0}\bar{m}. \tag{23}$$

Solutions are constant in time but not necessarily uniform in space, in case of heterogeneous solutions.

Let $U(t,x) = U(x) = (\rho(x), u(x), \omega(x))$ be a solution which holds the following system:

$$\begin{cases} \rho = m(x), & x \in \Gamma, \\ a.\nabla \rho = -\sigma\rho + \varphi(\rho)u, & x \in \Omega, \\ \dfrac{\partial u}{\partial t} = \beta\delta\omega - \gamma(\rho)u, & x \in \Omega, \\ \dfrac{\partial \omega}{\partial t} = d\Delta\omega - \beta\omega + \alpha(\rho)u, & x \in \Omega, \\ \dfrac{\partial \omega}{\partial \nu}(x) = 0, & x \in \partial\Omega. \end{cases} \tag{24}$$

Stability analysis

Stationary heterogeneous solution

For $(u, \omega) = (0, 0)$, system (24) admits one solution $U = (\rho, 0, 0)$ satisfying the stationary advection system and the expression of $\rho(x)$ is explicitly given by:

$$\rho(x) = m(x_0)e^{-s\sigma}, \quad (x_0, s) = (\zeta_1(x), \zeta_2(x)), \quad x = \xi(x_0, s) \in \Omega.$$

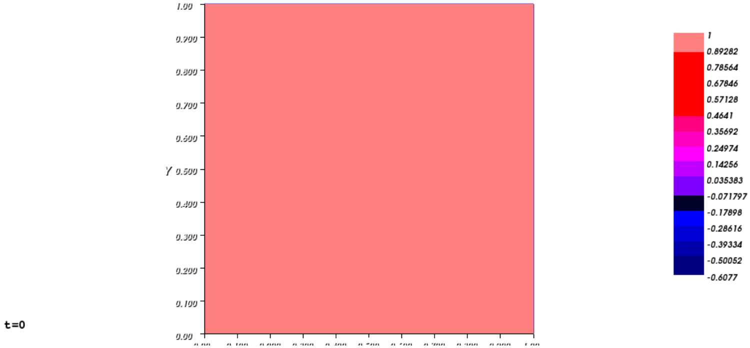


Figure 4 : Case of the non-existence of forest. It refers to the stationary homogeneous state $(0, 0, 0)$ of system (4).

Discussions

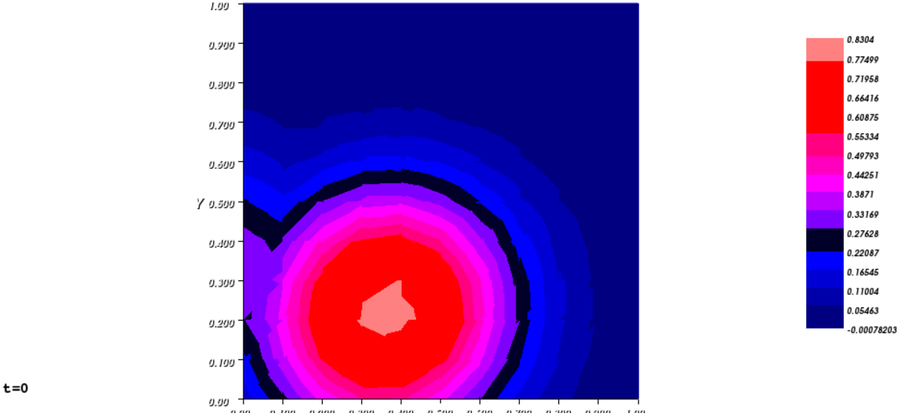


Figure 5 : $(\rho(x), 0, 0)$ after $t = 0$.

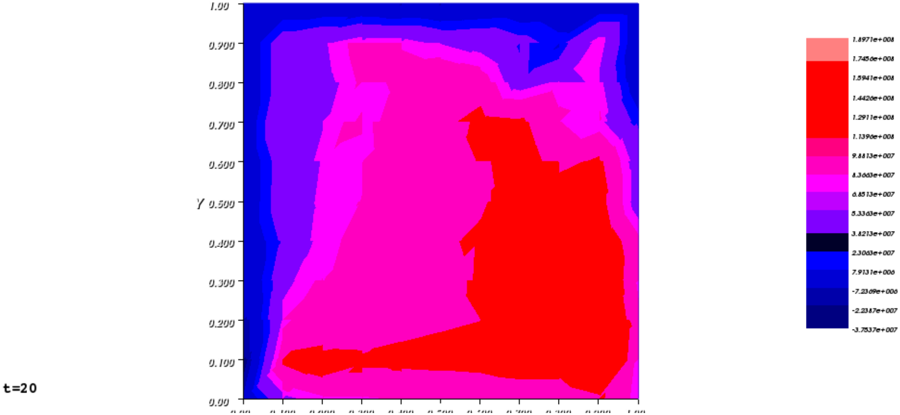


Figure 6 : $(\rho(x), 0, 0)$ after $t = 20$ years.

Conclusion & future plans

- We adapted the model proposed by Cantin et al.

- We highlighted the major impacts of some parameters in the forest dynamic.

- "Our model is well fitted".

Some references

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Thank you

For

Listening.